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MATHEMATICS

Limitations and Dependence on Parameter of Solutions of Non-stationary Almost Linear Differential Operators Equations

by

W. MLAK

Presented by T. WAŻEWSKI on August 13, 1958

We shall investigate the equation of the form

$$\frac{dx}{dt} = A(t)x + f(t, x), \tag{1}$$

where $A(t)$ is a closed linear operator defined on a linear subset of the Banach space E. The values of $A(t)$ belong to E. The theory of Eq. (1) is the continuation of the theory of semi-groups of linear bounded operations founded by Hille and Yosida [2]. A number of theorems concerning Eq. (1) have been presented in [5] and [6]. In this paper we investigate Eq. (1) by means of the method of differential inequalities. We use essentially the epidermic form of theorems about differential inequalities ([7] and [12]).

1. Let E be a Banach space. The elements of E are denoted by $x, y, z, \ldots$ Functions of the real variable t, with values lying in E are denoted by $x(t), y(t), z(t), \ldots$ etc. $|x|$ is the norm of the element x. The value of the functional ξ at the point x is denoted by ξx. We introduce the following assumptions:

Assumption (H_1). The function $\sigma(t, u)$ is continuous and non-negative for $t \in [0, a]$ and $u \geqslant 0$. Furthermore, for every $\eta \geqslant 0$ the right maximal solution $\omega(t, \eta)$ of the equation $u' = \sigma(t, u)$, passing through the point $(0, \eta)$ exists in the whole interval $[0, a]$.

Assumption (H_2). For every $t \in (0, a]$ the closed and linear operator $A(t)$ defined on a dense and linear subset of E possesses the resolvent $R(\lambda, A(t))$ for λ sufficiently large ($\lambda \geqslant \lambda(t)$).

Assumption (H_2^0). For every $t \in [0, a]$ the closed and linear operator $A(t)$ defined on a linear and dense subset of E possesses the resolvent $R(\lambda, A(t))$ for λ sufficiently large $(\lambda \geqslant \lambda(t))$.

We introduce the following notation

$$D_+ x(t) = \lim_{h \to 0+} \frac{x(t+h) - x(t)}{h}$$

if the limit on the right exists.

THEOREM 1. *Suppose that assumption* (H_1) *is satisfied. Let* $A_1(t)$ *and* $A_2(t)$ *satisfy the assumption* (H_2). *We assume that for every* $x \in E$ *the relation* $\lim_{\lambda \to +\infty} \lambda R(\lambda, A_i(t))x = x$ $(i = 1, 2;\ t \in (0, a])$ *holds. Suppose that the functions* $x_i(t)$ $(i = 1, 2)$ *are continuous in* $[0, a]$ *and satisfy the equations* $D_+ x_i(t) = A_i(t)x_i(t) + f_i(t, x_i(t))$ $(i = 1, 2)$ *except a subset of* $(0, a]$ *at most denumerable. We assume that for every triple* (x, y, t) $(x, y \in E,\ t \in (0, a])$ *there exists* $\lambda(x, y, t)$ *such that for* $\lambda \geqslant \lambda(x, y, t)$ *the inequality*

$$\left| \lambda[R(\lambda, A_1(t))x - R(\lambda, A_2(t))y] + \frac{1}{\lambda}[f_1(t, x) - f_2(t, y)] \right| \leqslant \frac{1}{\lambda}\sigma(t, |x-y|) + |x-y|$$

holds.

Under the assumptions given above the inequality

$$|x_1(t) - x_2(t)| \leqslant \omega(t, |x_1(0) - x_2(0)|)$$

holds for $t \in [0, a]$.

Theorem 1 implies the following uniqueness theorem analogous to Theorem 10 of [11].

THEOREM 2. *Suppose that the assumptions* (H_1) *and* (H_2) *are satisfied. We assume that* $|\lambda R(\lambda, A(t))| \leqslant 1$, $\omega(t, 0) \equiv 0$ *and* $|f(t, x) - f(t, y)| \leqslant \sigma(t, |x-y|)$ $(0 < t \leqslant a,\ x, y \in E)$. *Then, through every* $x_0 \in E$, *can pass at most one solution of the equation* $D_+ x(t) = A(t)x(t) + f(t, x(t))$, *continuous in* $[0, a]$ *and satisfying the equation in* $(0, a]$.

Remark that in Theorem 2 we do not assume that the equation is satisfied at the initial point $t = 0$. Theorems 1 and 2 are the generalization of Theorem 1 of Kato's paper [3].

THEOREM 3. *Suppose that the assumption* (H_1) *holds and* $A(t)$ *satisfies* (H_2). *Assume that* $|\lambda R(\lambda, A(t))| \leqslant 1$ *and* $|f(t, x)| \leqslant \sigma(t, |x|)$. *Let the continuous function* $x(t)$ *satisfy the equation* $D_+ x(t) = A(t)x(t) + f(t, x(t))$ *except an at most denumerable subset of* $(0, a]$. *Under our assumptions the inequality* $|x(t)| \leqslant \omega(t, |x(0)|)$ *holds for* $t \in [0, a]$.

The theorems given above permit to formulate general theorems concerning continuous dependence of solutions on the initial point and the right hand member for equations of the form (1).

By means of Theorem 3, applying the Leray-Schauder method, just as in [8], a non-local generalization of Theorem 7 of [6] can be proved. To simplify our considerations we assume that A does not depend on t. If A depends on t it suffices to introduce suitable conditions presented in [6]. We have the following non-local existence theorem:

THEOREM 4. *Let E be a Hilbert space and let A be a self-adjoint operator such that $(Ax, x) \leqslant -(x, x)(x \in D[A])$. Suppose that A^{-1} is completely continuous. We assume that the function $f(t, x)$ is continuous for $t \in [0, a]$ and $x \in E$ and satisfies the local Hölder's condition with regard to t and x for $t \in (0, a]$ and $x \in E$. Assume that $\sigma(t, u)$ satisfies (H_1). Suppose that $|f(t, x)| \leqslant \sigma(t, |x|)$ $(0 < t \leqslant a,\ x \in E)$. Under our assumptions for every $x_0 \in E$ there exists in the whole interval $(0, a]$ the continuous solution $x(t)$ of the equation $x' = Ax + f(t, x)$ satisfying the initial condition $\lim_{t \to 0+} x(t) = x_0$.*

In the proof of Theorem 4 we use essentially the results presented in [6].

If $f(t, x)$ satisfies the Hölders condition

$$|f(\bar{t}, \bar{y}) - f(\bar{\bar{t}}, \bar{\bar{y}})| \leqslant K[|\bar{t} - \bar{\bar{t}}|^{\gamma} + |\bar{y} - \bar{\bar{y}}|^{\gamma}] \qquad (0 < \gamma \leqslant 1)$$

we put $\sigma(t, u) = au + b$, where a and b are suitable constants such that $\max_{0 \leqslant s \leqslant a} |f(s, 0)| + Ka^{\gamma} + K|u| \leqslant a|u|^{\gamma} + b$.

We have the following uniqueness theorem corresponding to the general theorem of Kamke [13].

THEOREM 5. *Let $A(t)$ satisfy (H_2^0). Suppose that $|\lambda R(\lambda, A(t))| \leqslant 1$. Assume that the non-negative function $\sigma(t, u)$ is continuous for $0 < t < a$ and $u \geqslant 0$. Suppose that for every $\varrho \in (0, a)$ the unique solution $\omega(t)$ of the equation $u' = \sigma(t, u)$, valid in the interval $(0, \varrho)$ and satisfying the conditions $\lim_{t \to 0+} \omega(t) = \lim_{t \to 0+} \omega'(t) = 0$, is the solution identically equal to zero in the interval $(0, \varrho)$. We assume that $|f(t, x) - f(t, y)| \leqslant \sigma(t, |x - y|)$ for $0 < t \leqslant a$ and $x, y \in E$. These assumptions imply that through every point $(0, x)$ $(x \in E)$ can pass at most one solution of the equation $x' = A(t)x + f(t, x)$ for $t \in [0, a]$.*

Take now the functional ξ and suppose that ξ has the Fréchet differential which we denote by $L(x, h)$. We have the following theorem:

THEOREM 6. *Let the assumptions (H_1) and (H_2) be satisfied. Assume that $\lim_{\lambda \to \infty} \lambda R(\lambda, A(t))x = x$ for $t \in (0, a]$ and $x \in E$. Suppose that $L(x, \lambda R(\lambda, A(t))x) \leqslant L(x, x)$ for $\lambda > 0$ and $L(x, f(t, x)) \leqslant \sigma(t, \xi x)$. We assume that the continuous function $x(t)$ satisfies the equation $D_+ x(t) = A(t)x(t) + f(t, x(t))$ except a subset of $(0, a]$ at most denumerable. Under our assumptions the inequality $\xi x(t) \leqslant \omega(t, \xi x(0))$ for $t \in [0, a]$ holds.*

2. We shall investigate now the generalized solutions. To begin with, we introduce the assumptions of Kato [3], [5].

Assumptions (H_3). Let $A(t)$ satisfy (H_3^0). We assume that the set in which $A(t)$ is defined does not depend on t, i. e. $D[A(t)] = D = \text{const}$. Furthermore, we assume that $|R(\lambda, A(t))| \leqslant \frac{1}{\lambda+1}$ for $\lambda > -1$ and $t \in [0, a]$ and that the bounded strong derivative $\frac{\partial}{\partial t} A(t) A^{-1}(s)$ exists for $t, s \in [0, a]$ and is strongly continuous in t for fixed s.

Assumptions (H_3) imply the existence of the strongly continuous operators function $U(t, s)$ such that $U(s, s) = I$. For $x \in D$ the function $x(t) = U(t, s)x$ satisfies the equation $x' = A(t)x$. The continuous function $y(t)$ is called the generalized solution of Eq. (1), if it satisfies the integral equation

$$y(t) = U(t, 0)y(0) + \int_0^t U(t, s) f(s, y(s))\, ds .$$

THEOREM 7. *Let the assumptions* (H_1) *and* (H_3) *be satisfied. The function* $x_i(t)$ $(i = 1, 2)$ *is the generalized solution of the equation* $x = A(t)x + f_i(t, x)$ $(i = 1, 2)$. *Suppose that* $|f_1(t, x) - f_2(t, y)| \leqslant \sigma(t, |x - y|)$. *We assume additionally that* $\sigma(t, u)$ *increases in* u. *Assumptions given above imply the inequality* $|x_1(t) - x_2(t)| \leqslant \omega(t, |x_1(0) - x_2(0)|)$, $t \in [0, a]$.

THEOREM 8. *Let the assumptions* (H_1) *and* (H_3) *be satisfied. Assume that* $\sigma(t, u)$ *increases in* u. *The function* $x(t)$ *is the generalized solution of the equation* $x' = A(t)x + f(t, x)$. *Suppose that* $|f(t, x)| \leqslant \sigma(t, |x|)$. *Our assumptions imply the inequality* $|x(t)| \leqslant \omega(t, |x(0)|)$ $(0 \leqslant t \leqslant a)$.

In the proofs of Theorems 7 and 8 we use a theorem on integral inequalities (cf. [10], Theorem 1). By means of Theorem 8, introducing the norm proposed by A. Bielecki in [1] for the space $C_E[0, a]$, applying the properties of the resolvent of the non-linear operators (see [4], p. 148) and using the Leray-Schauder method the following non-local existence theorem (see [5], Theorem 5) can be proved.

THEOREM 9. *Let the assumptions of Theorem* 8 *be satisfied. We assume that* $f(t, x) = h(t, x) + g(t, x)$, *where* $h(t, x)$ *is continuous and satisfies the Lipschitz condition* $|h(t, x) - h(t, y)| \leqslant R(t)|x - y|$ *with summable* $R(t)$ *and* $g(t, x)$, *is completely continuous. Under our assumptions for every* $x_0 \in E$ *there exists in* $[0, a]$ *the continuous solution* $x(t)$ *of the equation* $x(t) = U(t, 0)x_0 + \int_0^t U(t, s) f(s, x(s))\, ds$.

The theorems of paragraphs 1. and 2. may be generalized if we apply theorems concerning finite systems of differential inequalities [11] just as it was done in [8]. In this case instead of the norm we take a finite

sequence of pseudo-norms. Similar investigations may be done for finite systems of functionals possessing Fréchet differentials.

3. We now investigate the dependence on parameter. We introduce the following assumption:

Assumption (H_4). We assume that for every fixed $\mu \in [\underline{\mu}, \overline{\mu}]$ $(\underline{\mu} < \overline{\mu})$ the operator $A(t, \mu)$ satisfies (H_3) for $t \in [0, a]$. The set $D[A(t, \mu)]$ does not depend on t and μ, i. e. $D[A(t, \mu)] = D = \text{const}$. We suppose that for every $x \in E$, $t \in [0, a]$ and $\mu, \lambda \in [\underline{\mu}, \overline{\mu}]$ there exists the limit

$$\lim_{h \to 0} \frac{A(t, \mu+h)A^{-1}(t, \lambda)x - A(t, \mu)A^{-1}(t, \lambda)x}{h} = B(t, \mu, \lambda)x$$

and

$$|B(t, \mu, \lambda)| \leqslant M \quad (M = \text{const} < +\infty) \qquad \text{for} \qquad t \in [0, a], \ \mu, \lambda \in [\underline{\mu}, \overline{\mu}].$$

The operation $B(t, \mu, \lambda)$ is supposed to be strongly continuous in t.

Denote by $U(t, s, \lambda)$ Kato's function corresponding to the operator $A(t, \lambda)$. We have the following theorems:

Theorem 10. *Suppose that the assumption* (H_4) *holds. Take* $x \in D$. *Then*

$$\lim_{h \to 0} U(t, s, \lambda+h)x = U(t, s, \lambda)x$$

uniformly with regard to t *for fixed* s, *and there exists the derivative* $\dfrac{\partial U(t, s, \lambda)x}{\partial \lambda}$ *given by the formula*

$$\frac{\partial U(t, s, \lambda)x}{\partial \lambda} = \int_s^t U(t, \tau, \lambda)B(\tau, \lambda, \lambda)A(\tau, \lambda)U(\tau, s, \lambda)x\,d\tau.$$

Theorem 11. *Suppose that the assumption* (H_4) *holds. We assume that the continuous function* $f(t, y, \lambda)$ *possesses the bounded and continuous derivative* $\dfrac{\partial f}{\partial \lambda}$ *and the bounded and continuous Fréchet differential* $L(t, x, \lambda)$ *taken with respect to* y. *Let* $z(t, \lambda)$ *be the continuously differentiable in* t *solution of the equation* $x' = A(t, \lambda)x + f(t, x, \lambda)$. *Assume that* $z(0, \lambda) = \text{const} = x \in D$ *for* $\lambda \in [\underline{\mu}, \overline{\mu}]$. *Under the assumptions given above there exists the derivative* $\dfrac{\partial z}{\partial \lambda}$, *continuous in* t *and satisfying the integral equation*

$$\begin{aligned}\frac{\partial z(t, \lambda)}{\partial \lambda} = &\int_0^t U(t, \tau, \lambda)B(\tau, \lambda, \lambda)\left[\frac{\partial z(t, \lambda)}{\partial t}\right]_{t=\tau} d\tau + \\ &+ \int_0^t U(t, \tau, \lambda)\left[\frac{\partial f(\tau, x, \lambda)}{\partial \lambda}\right]_{x=z(\tau, \lambda)} d\tau + \\ &+ \int_0^t U(t, \tau, \lambda)L(\tau; z(\tau, \lambda), \lambda)\frac{\partial z(\tau, \lambda)}{\partial \lambda}\,d\tau.\end{aligned}$$

In the proofs of Theorems 10 and 11 we use the formula (12) of [5].

Results given in this paper were presented at the session of the Polish Mathematical Society in Warsaw on May 30, 1958.

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REFERENCES

[1] A. Bielecki, *Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires*, Bull. Acad. Polon. Sci. Cl. III, **4** (1956), 261.

[2] E. Hille, *Functional analysis and semigroups*, New York, 1948.

[3] T. Kato, *Integration of the equation of evolution in Banach space*, Journal of Math. Soc. of Japan **5** (1953), 208-234.

[4] M. A. Krasnoselsky, *Topological methods in the theory of non-linear integral equations* (in Russian), Moscow, 1956.

[5] M. A. Krasnoselsky, S. G. Krein and P. E. Sobolewsky, *On differential equations with unbounded operators in Banach spaces* (in Russian), D. A. N. USSR **111** (1956), 19-22.

[6] — *On differential equations with unbounded operators in Hilbert spaces* (in Russian), D. A. N. USSR **112** (1957), 990-993.

[7] W. Mlak, *On the epidermic effect for ordinary differential inequalities of the first order*, Annales Polon. Math. **3** (1956), 37-40.

[8] — *A note on non-local existence of solutions of ordinary differential equations*, Annales Polon. Mat. **4** (1958), 344-347.

[9] J. Leray et J. Schauder, *Topologie et équations fonctionnelles*, Annales de l'Ecole Normale Sup. **13** (1934), 45-78.

[10] Z. Opial, *Sur un système d'inégalités intégrales*, Annales Polon. Math. **3** (1957), 200-209.

[11] T. Ważewski, *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*, Annales Soc. Math. Pol. **23** (1950), 112-156.

[12] — *Certaines propositions de caractère épidermique relatives aux inégalités différentielles*, Annales Soc. Pol. Math. **24** (1951), 1-12.

[13] E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig, 1930.

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MATHÉMATIQUE

Sur la méthode des approximations successives pour un système de n équations à n inconnues

par

J. KISYŃSKI

Présenté par S. MAZUR le 16 Septembre 1958

Nous avons donné en collaboration avec M. A. Bielecki [1] une démonstration plus rapide d'un intéressant lemme due à Mlle Z. Szmydt [4]. Cette démonstration m'a suggéré une méthode qui permet d'établir les théorèmes suivants:

THÉORÈME 1. *Si*

(i) *F est une transformation du produit cartésien $E = E_1 \times E_2 \times \ldots \times E_n$ d'espaces métriques complets en lui-même, satisfaisant aux conditions de Lipschitz*

$$d_i(y_i', y_i'') \leqslant \sum_{k=1}^{n} L_{ik} d_k(x_k', x_k''), \qquad i = 1, 2, \ldots, n, \tag{1}$$

pour $x' \in E$ et $x'' \in E$ quelconques où

$$(y_1', y_2', \ldots, y_n') = y' = F(x'),\ x' = (x_1', x_2', \ldots, x_n'),$$

$$(y_1'', y_2'', \ldots, y_n'') = y'' = F(x''),\ x'' = (x_1'', x_2'', \ldots, x_n''),$$

d_i désigne la distance dans l'espace, E_i et L_{ik} sont des constantes non négatives, indépendantes de x' et x''

et

(ii) *tous les mineurs principaux de la matrice $\|\delta_{ik} - L_{ik}\|_1^n$ sont positifs,*

alors

il existe dans l'espace E exactement un élément $\overset{0}{x}$ tel que $F(\overset{0}{x}) = \overset{0}{x}$ et, de plus, pour $x \in E$ quelconque,

$$\lim_{m\to\infty} d_i(\overset{0}{x_i}, \overset{m}{x_i}) = 0 \qquad \text{pour} \qquad i = 1, 2, \ldots, n,$$

où

$$(\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_n) = \overset{0}{x}$$

et

$$(\overset{m}{x}_1, \overset{m}{x}_2, \ldots, \overset{m}{x}_n) = \overset{m}{x}$$

où

$$\overset{1}{x} = F(x), \quad \overset{m+1}{x} = F(\overset{m}{x}) \quad \text{pour} \quad m = 1, 2, \ldots .$$

THÉORÈME 2. *Si la matrice* $\|L_{ik}\|_1^n$, *dont les éléments sont non négatifs, est telle que pour toute transformation* F *ayant la propriété* (i) *il existe un élément* $\overset{0}{x} \in E$ *tel que* $F(\overset{0}{x}) = \overset{0}{x}$, *alors la condition* (ii) *est satisfaite.*

Il convient de remarquer que l'on a les deux théorèmes suivants:

THÉORÈME 3 (de Kotelianski). *Si* $L_{ik} \geqslant 0$ *pour* $i, k = 1, 2, \ldots, n$ *et* $|\lambda\delta_{ik} - L_{ik}|_1^p > 0$ *pour* $p = 1, 2, \ldots, n$, *tous les mineurs principaux de la matrice* $\|\lambda\delta_{ik} - L_{ik}\|_1^n$ *sont positifs* *).

THÉORÈME 4. *Si* $L_{ik} \geqslant 0$ *pour* $i, k = 1, 2, \ldots, n$, $L_{ii} < \lambda$ *pour* $i = 1, 2, \ldots, n$ *et* $|\lambda\delta_{r_i r_k} - L_{r_i r_k}|_1^p > 0$ *pour tout système d'indices* $r_1, r_2, \ldots, r_p$ *tel que* $n \geqslant p \equiv n \pmod 2$ *et* $1 \leqslant r_1 < r_2 < \ldots < r_p \leqslant \frac{n+p}{2}$, *alors tous les mineurs principaux de la matrice* $\|\lambda\delta_{ik} - L_{ik}\|_1^n$ *sont positifs.*

Nous allons démontrer les théorèmes 1, 2 et 4. Dans ce but considérons la matrice $A = \|a_{ik}\|_1^n$ et, pour un partage quelconque des nombres $1, 2, \ldots, n$ en classes disjointes $p_1, p_2, \ldots, p_\nu$ et $r_1, r_2, \ldots, r_\mu$ et pour un partage quelconque de ces nombres en classes disjointes $q_1, q_2, \ldots, q_\nu$ et $s_1, s_2, \ldots, s_\mu$, posons

$$A_{s_1 s_2 \ldots s_\mu}^{r_1 r_2 \ldots r_\mu} = A\begin{pmatrix} p_1 p_2 \ldots p_\nu \\ q_1 q_2 \ldots q_\nu \end{pmatrix} = |a_{p_i q_k}|_1^\nu .$$

Désignons encore par $p_{(q_1 q_2 \ldots q_s)}$ le nombre des entiers positifs non supérieurs à p et différents de chacun des nombres $q_1, q_2, \ldots, q_s$. En profitant des égalités évidentes

$$i_{(k)} + k_{(i)} = i + k - 1 \quad \text{pour} \quad i \neq k ,$$

$$k + i + l_{(ik)} = \begin{cases} l_{(i)} + i_{(k)} + k_{(il)} , & \text{si } k < l \\ l_{(i)} + i_{(k)} + k_{(il)} + 2 , & \text{si } k > l \end{cases} \quad \text{pour} \quad k \neq i \neq l ,$$

on tire des formules

$$A_k^i = \sum_{\substack{l=1,2,\ldots,n \\ l \neq i}} (-1)^{l_{(i)} + i_{(k)}} a_{li} A_{ki}^{il} \quad \text{pour} \quad i \neq k ,$$

*) Cette dernière condition est équivalente à ce que la plus grande racine caractéristique positive de la matrice $\|L_{ik}\|_1^n$ soit inférieure à λ (cf. [2], p. 337).

$$A_{ki}^{il} = \sum_{\substack{m=1,2,\dots,n \\ i \neq m \neq k}} (-1)^{k(il)+m(ik)} a_{km} A_{kim}^{ilk} \quad \text{pour} \quad i \neq k,\ i \neq l \neq k,$$

la formule suivante

$$(2) \quad (-1)^{i+k} A_k^i = a_{ki} A_{ik}^{ik} + \sum_{\substack{l,m=1,2,\dots,n \\ i \neq l \neq k \\ i \neq m \neq k}} (-1)^{l(ik)+m(ik)} a_{li} a_{km} A_{ikm}^{ikl} \quad \text{pour} \quad i \neq k\ ^*).$$

Moyennant cette formule, en procédant par récurrence séparément pour n pair et n impair, on obtient aisément le lemme suivant:

LEMME 1. *Si dans la matrice $\|a_{ik}\|_1^n$ tous les éléments situés en dehors de la diagonale principale sont non positifs et si tous ses mineurs principaux d'ordre p, où $n > p \equiv n \pmod 2$, sont non négatifs, alors $(-1)^{i+k} A_k^i \geqslant 0$ pour $i \neq k$, où $i, k = 1, 2, \dots, n$.*

LEMME 2. *Pour la matrice $\|a_{ik}\|_1^n$ dont tous les éléments situés en dehors de la diagonale principale sont non positifs, les conditions suivantes sont équivalentes:*

(A) *il existe un système de nombres positifs $x_1, x_2, \dots, x_n$ tel que*

$$\sum_{k=1}^{n} a_{ik} x_k > 0 \quad \text{pour} \quad i = 1, 2, \dots, n;$$

(B) *tous les mineurs principaux de la matrice $\|a_{ik}\|_1^n$ sont positifs;*

(C) $$A \begin{pmatrix} 12 \dots p \\ 12 \dots p \end{pmatrix} > 0 \quad \text{pour} \quad p = 1, 2, \dots, n;$$

(D) *$a_{ii} > 0$ pour $i = 1, 2, \dots, n$ et $A \begin{pmatrix} r_1 r_2 \dots r_p \\ r_1 r_2 \dots r_p \end{pmatrix} > 0$ pour tout système d'indices $r_1, r_2, \dots, r_p$ tel que $n \geqslant p \equiv n \pmod 2$ et $1 \leqslant r_1 < r_2 < \dots < r_p \leqslant \frac{n+p}{2}$.*

Démonstration de l'équivalence (A) $\rightleftarrows$ (B). En posant $x_k = \sum_{i=1}^{n} (-1)^{i+k} A_k^i$ et en profitant du lemme 1 on voit que (B)$\rightarrow$(A). L'implication inverse sera établie par récurrence. Pour $n = 1$ on a evidemment (A)$\rightarrow$(B). Supposons que l'on ait (A)$\rightarrow$(B) pour $n = p - 1$ et admettons qu'une matrice $\|a_{ik}\|_1^p$ satisfait à la condition (A) pour $n = p$. Les inégalités

$$\sum_{k=1}^{m-1} a_{ik} x_k + \sum_{k=m+1}^{p} a_{ik} x_k > 0, \quad i = 1, 2, \dots, m-1, m+1, \dots, p$$

sont alors vérifiées pour tout $m = 1, 2, \dots, p$, donc, en vertu de l'hypothèse de récurrence, tous les mineurs principaux d'ordre $< p$ de la ma-

*) Cette formule, de même que le lemme 1, sont bien connus (cf. p. ex. [3]). Nous en avons rappelé la preuve pour rendre plus claires les considérations qui suivent.

trice $\|a_{ik}\|_1^p$ sont positifs (en particulier $A_p^p > 0$); par conséquent on a, en vertu du lemme 1,

$$|a_{ik}|_1^p = x_p^{-1} \cdot \sum_{i,k=1}^{p} (-1)^{i+p} A_p^i a_{ik} x_k > 0$$

et par suite la matrice $\|a_{ik}\|_1^p$ satisfait à la condition (B) pour $n = p$.

La démonstration de l'équivalence (B) $\equiv$ (C), due à Kotelianski, est exposée dans la monographie [2], p. 338, cf. aussi [3].

Démonstration de l'équivalence (B) $\equiv$ (D). Pour les matrices d'ordre 1 et 2 les conditions (B) et (D) désignent la même chose. Supposons qu'elles soient équivalentes pour $n = p-2$ et admettons qu'une matrice $\|a_{ik}\|_1^p$ satisfait à la condition (D) pour $n = p$. Pour tout $s = 1, 2, \ldots, p-1$ et pour tout système d'indices $r_1, r_2, \ldots, r_q$ tel que $r_\nu \neq s$, $\nu = 1, 2, \ldots, q$, $p-2 \geqslant q \equiv p \pmod 2$ et

$$1 \leqslant r_1 < r_2 < \ldots < r_q \leqslant \begin{cases} \dfrac{p-2+q}{2}, & \text{si} \quad r_q < s, \\ \dfrac{p+q}{2}, & \text{si} \quad r_q > s, \end{cases}$$

l'inégalité $A\begin{pmatrix} r_1 r_2 \ldots r_q \\ r_1 r_2 \ldots r_q \end{pmatrix} > 0$ est alors vérifiée, d'où il résulte que si dans une quelconque des matrices $\|a_{ik}\|$, $i \neq s \neq k$, $i, k = 1, 2, \ldots, p-1$, $s = 1, 2, \ldots, p-1$, tout en ne changeant pas l'ordre des lignes et des colonnes, on leur assigne les numéros $1, 2, \ldots, p-2$, une telle matrice devra aussi satisfaire à la condition (D) pour $n = p-2$. Donc, en vertu de l'hypothèse de récurrence, les inégalités $A\begin{pmatrix} r_1 r_2 \ldots r_q \\ r_1 r_2 \ldots r_q \end{pmatrix} > 0$ sont vérifiées pour tout système d'indices $r_1, r_2, \ldots, r_q$ tel que $1 \leqslant r_1 < r_2 < \ldots < r_q \leqslant p-1$, $q \leqslant p-2$. Par conséquent, en vertu du lemme 1, on a $(-1)^{l(k)+m(k)} A_{pkm}^{pkl} \geqslant 0$ pour $l \neq k \neq m$, $k, l, m = 1, 2, \ldots, p-1$, d'où on tire, d'après la formule (2) $(-1)^{p+k} A_k^p \geqslant 0$ pour $k = 1, 2, \ldots, p-1$, donc $A_p^p = a_{pp}^{-1} \cdot \left(|a_{ik}|_1^p - \sum_{k=1}^{p-1} (-1)^{p+k} a_{pk} A_k^p\right) \geqslant a_{pp}^{-1} \cdot |a_{ik}|_1^p > 0$ et, par suite, la matrice $\|a_{ik}\|_1^p$ satisfait à la condition (C) pour $n = p$. Pour achever la démonstration il suffit d'appliquer l'équivalence (C) $\equiv$ (B).

Les théorèmes 3 et 4 résultent immédiatement des équivalences (B) $\equiv$ (C) et (B) $\equiv$ (D).

Démonstration du théorème 1. Si les hypothèses (i) et (ii) sont vérifiées, il existe, en vertu de l'équivalence (A) $\equiv$ (B), un système de nombres positifs $a_1, a_2, \ldots, a_n$ tel que $\sum_{i=1}^{n} (\delta_{ik} - L_{ik}) a_i > 0$ pour $k = 1, 2, \ldots, n$,

donc il existe un λ tel que

$$0 < \lambda < 1 \quad \text{et} \quad \sum_{i=1}^{n} L_{ik} a_i \leqslant \lambda a_k \quad \text{pour} \quad k = 1, 2, \ldots, n .$$

En introduisant dans l'espace E la distance

$$d(x', x'') = \sum_{i=1}^{n} a_i d_i(x_i', x_i'')$$

nous obtenons un espace métrique complet dans lequel, en vertu de (1), on a, pour $x' \in E$ et $x'' \in E$ quelconques, l'inégalité

$$d(F(x'), F(x'')) \leqslant \lambda d(x', x'') ;$$

le théorème 1 résulte alors du théorème bien connu du point fixe de Banach.

On arrive au même résultat en introduisant dans l'espace E la distance

$$d(x', x'') = \max_{i=1,2,\ldots,n} \beta_i^{-1} d_i(x_i', x_i'') ,$$

où $\beta_1, \beta_2, \ldots, \beta_n$ sont des nombres positifs tels que

$$\sum_{k=1}^{n} (\delta_{ik} - L_{ik}) \beta_k > 0 \quad \text{pour} \quad i = 1, 2, \ldots, n .$$

Démonstration du théorème 2. Prenons pour E l'espace euclidien à n dimensions et considérons la transformation

$$y_i = 1 + \sum_{k=1}^{n} L_{ik} |x_k| , \qquad i = 1, 2, \ldots, n .$$

Les conditions de Lipschitz (1) sont alors remplies et, par conséquent, il existe un point fixe de cette transformation. Il existe donc un système de nombres positifs $\overset{0}{x}_1, \overset{0}{x}_2, \ldots, \overset{0}{x}_n$ tel que

$$\sum_{k=1}^{n} (\delta_{ik} - L_{ik}) \overset{0}{x}_k = 1 \quad \text{pour} \quad i = 1, 2, \ldots, n$$

et, pour achever la démonstration, il suffit d'appliquer l'équivalence (A) $\equiv$ (B).

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OUVRAGES CITÉS

[1] A. Bielecki et J. Kisyński, *Une remarque à propos de deux notes de Z. Szmydt*. Bull. Acad. Polon. Sci., Série des sci. math., astr. et phys., **6** (1958), 15.

[2] F. R. Gantmaher. *La théorie des matrices* (en russe), Gos. Izd. Techn.-Teor. Lit., Moscow, 1933.

[3] D. M. Kotelianskii, *Sur quelques propriétés des matrices avec elements positifs* (en russe), Mat. Sbor., **31** (1952), 497.

[4] Z. Szmydt, *Sur le problème de Goursat concernant les equations differentielles hyperboliques du second ordre*, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 572.

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MATHEMATICS

Formula for the Spectral Decomposition

by

T. LEŻAŃSKI

Presented by S. MAZUR on September 16, 1958

Let H be a complex Hilbert space, A — a bounded symmetric operation defined on it, whose spectrum is wholly contained within a finite segment $\Delta = (a, b)$. The spectral family of operations A will be denoted by E_λ. Finally, let z^0 be a fixed element of space H. The aim of the present note is to give an effective formula for the abstract function $E_\lambda(z^0)$.

Accordingly, let us introduce a space $\mathcal{H}$ of functions defined on segment Δ, with values in H, for which $\int_\Delta \|x(\lambda)\|^2 d\lambda < +\infty$. Defining the scalar product in $\mathcal{H}$ by the formula

$$\big(x(\cdot), y(\cdot)\big) = \int_\Delta \big(x(\lambda), y(\lambda)\big)\, d\lambda\,,$$

we shall obtain a real Hilbert space.

Denote by $\mathfrak{X}_1$ the linear space of abstract functions of the form $y(\lambda) = \int_a^\lambda x(\mu)\, d\mu$, where $x(\cdot) \in \mathcal{H}$. In space $\mathfrak{X}_1$ we introduce pseudonorms as follows: let Z be the set of real functions satisfying the Lipschitz condition and let

$$\|y(\cdot)\|_\psi \overset{\text{df}}{=} \Big\| \int_\Delta \psi(\lambda) x(\lambda)\, d \Big\| \quad \text{if} \quad y(\lambda) = \int_a^\lambda x(\mu)\, d\mu \quad \text{and} \quad \psi \in Z\,.$$

Denote by $\mathfrak{X}$ the complement to space $\mathfrak{X}_1$ with the above mentioned pseudonorms and let the resulting convergence be termed convergence $(\mathfrak{m})$. The abstract function $E_\lambda(z^0)$ will be represented as the limit in the sense of $(\mathfrak{m})$ of the sequence of elements of space $\mathfrak{X}_1$. Accordingly, let us introduce two functionals defined on $\mathcal{H}$:

$$\Phi\big(x(\cdot)\big) = \int_\Delta \Big\| Ax(\lambda) - \lambda x(\lambda) \Big\|^2 d\lambda\,, \quad \Psi\big(x(\cdot)\big) = \Big\| \int_\Delta x(\lambda)\, d\lambda - z^0 \Big\|^2$$

and denote their sum by φ.

LEMMA 1. *If* $\varphi(x_n(\cdot))\to 0$ *and* $y_n(\lambda)=\int_a^\lambda x_n(\mu)\,d\mu$, *then* $y_n(\lambda)\to E_\lambda(z^0)$ *in the sense of* (m).

The functional φ is square, convex and continuous, and thus possesses a gradient — grad $\varphi(x(\cdot))$.

LEMMA 2. *The abstract function* $x_t(\lambda)$ *satisfying the differential equation:* $\frac{d}{dt}x_t(\lambda)=-\operatorname{grad}\varphi(x_t(\lambda))$, *is such that* $\varphi(x_t(\lambda))\to 0$ $(t\to\infty)$.

Let $\mathcal{U}(x(\lambda))=\{Ax(\lambda)-\lambda x(\lambda)\}$; let $\mathcal{V}(x(\lambda))=\{\int_\Delta x(\lambda)\,d\lambda\}$ be a operation with a constant value and finally $x^0(\lambda)=2\{z^0\}$ a function abstract with a constant value.

Let $\mathcal{A}(x(\lambda))=2\{\mathcal{U}^2(x(\lambda))+\mathcal{V}(x(\lambda))\}$.

LEMMA 3. *The abstract function defined by formula*

$$x_t(\lambda)=\int_0^t e^{-s\mathcal{A}}(x^0(\lambda))\,ds$$

satisfies the operation $\frac{d}{dt}x_t(\lambda)=-\operatorname{grad}\varphi(x_t(\lambda))$.

Summing up Lemmas 1., 2., and 3., we find that

$$\int_0^t\int_a^\lambda e^{-s\mathcal{A}}(x(\mu))\,d\mu\,ds\to E_\lambda(z^0)$$

in the sense of (m), which can be expressed as

THEOREM 1.

$$E_\lambda(z^0)=\int_0^\infty\int_a^\lambda e^{-s\mathcal{A}}(x^0(\mu))\,d\mu\,ds\,,$$

where the integral on the right is convergent in the sense of (m).

Let $A_1, A_2, \ldots A$ be bounded symmetric operations and let $\|A_n-A\|\to 0$ $(n\to\infty)$. Thus it may be assumed that the spectra of the operations $A_1, A_2, \ldots A$ are contained within the common segment $\Delta=(a,b)$.

Denote by $E_{n,\lambda}$ and E_λ the spectral families of the operations An and A, respectively, and let z denote a fixed element of space H. Under these assumptions we have

THEOREM 2. *If* $\|A_n-A\|\to 0$, *then*

$$\Big\|\int_\Delta \psi(\lambda)\,d_\lambda E_{n,\lambda}(z^0)-\int_\Delta \psi(\lambda)\,d_\lambda E_\lambda(z^0)\Big\|\to 0$$

for every function $\psi(\lambda)$ *satisfying the Lipschitz condition.*

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MATHÉMATIQUE

Sur la fonction dont la transformée de Laplace est e^{-s^α}

par

J. MIKUSIŃSKI

Présenté par A. MOSTOWSKI le 16 Septembre 1958

Dans la théorie de l'équation de la chaleur

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

la fonction suivante joue un rôle fondamental:

$$F(t) = \frac{1}{2\sqrt{\pi t^3}} e^{-1/4t} \qquad (t > 0)\,. \tag{1}$$

La question s'impose de trouver une fonction analogue pour l'équation plus générale

$$\frac{\partial^{2n} v}{\partial x^{2n}} = \frac{\partial v}{\partial t} \qquad (n \text{ naturel})\,. \tag{2}$$

La transformée de Laplace de cette équation est, les conditions sur l'axe x étant supposées nulles,

$$\frac{d^{2n} V}{dx^{2n}} = sv\,,$$

où V est la transformée de v. La dernière équation a deux solutions réelles

$$V = \exp(-xs^{1/2n}) \quad \text{et} \quad V = \exp(xs^{1/2n})\,. \tag{3}$$

La solution de (2) qui correspond à la première des fonctions (3) peut être obtenue par la formule bien connue d'inversion; on a ainsi

$$v = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{ty - xz^{1/2n}}\, dz\,. \tag{4}$$

Il n'existe cependant pas de solution de (2) correspondant à la seconde des fonctions (3).

La fonction (4) a été l'objet de recherches de plusieurs auteurs [1]-[6]. Elle est une fonction de deux variables $u = u(x, t)$ (dépendant en outre du paramètre $1/2n$), mais il suffit de l'étudier pour $x = 1$, parce que l'on peut revenir à sa forme générale par un simple changement des variables.

D'autre part, il semble naturel de remplacer le paramètre $1/2n$ par un paramètre continu, soit α. On est ainsi amené à considérer la fonction

$$F_\alpha(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-tz-z^\alpha}\, dz \qquad (0 < \alpha < 1)\,. \tag{5}$$

Pour $\alpha = \frac{1}{2}$, l'intégrale se laisse évaluer effectivement, ce qui conduit à la formule (1). On connaît aussi d'autres formules pour $F_\alpha(t)$, par exemple

$$F_\alpha(t) = \frac{1}{\pi} \int_0^\infty \exp(-tr - \gamma_1 v^\alpha) \sin(\sigma_1 v^\alpha)\, dr\,,$$

$$F_\alpha(t) = \frac{2}{\pi} \int_0^\infty \exp(-\gamma_2 r^\alpha) \sin(\sigma_2 r^\alpha) \sin tr\, dr\,,$$

$$F_\alpha(t) = \frac{2}{\pi} \int_0^\infty \exp(-\gamma_2 r^\alpha) \cos(\gamma_2 r^\alpha) \cos tr\, dr\,,$$

où

$$\gamma_1 = \cos \pi\alpha, \qquad \sigma_1 = \sin \pi\alpha, \qquad \gamma_2 = \cos \frac{\pi\alpha}{2}, \qquad \sigma_2 = \sin \frac{\pi\alpha}{2}\,.$$

Or, il est difficile de déduire de ces formules, comment se comporte $F_\alpha(t)$ dans l'interval $0 < t < \infty$, car les fonctions à intégrer sont oscillantes. Pareillement, l'étude de la formule (5) est difficile, car la fonction à intégrer est complexe. Pour cette raison nous nous sommes proposés de chercher une formule nouvelle qui serait plus commode dans les applications.

On peut changer le chemin d'intégration dans (5) de manière à obtenir une fonction réelle sous le signe d'intégrale. En effet, la fonction

$$e^{-tz-z^\alpha}$$

est réelle le long de la courbe

$$r = \left(\frac{\sin \alpha\varphi}{t \sin \varphi}\right)^{1/(1-\alpha)} \qquad \left(z = r(\cos\varphi + i \sin\varphi)\right)\,. \tag{6}$$

On peut démontrer, par les méthodes habituelles, que l'intégrale le long de la courbe (6) est égale à l'intégrale prise le long de l'axe imaginaire, c'est-à-dire à (5). En introduisant la variable φ au lieu de z, on

trouve ainsi que

$$F_\alpha(t) = \frac{1}{\pi}\,\frac{\alpha}{1-\alpha}\,\frac{1}{t}\int_0^\pi u e^{-u}\,d\varphi\,, \tag{7}$$

où

$$u = t^{-\alpha/(1-\alpha)}\left(\frac{\sin\alpha\varphi}{\sin\varphi}\right)^{1/(1-\alpha)}\frac{\sin(1-\alpha)\varphi}{\sin\varphi}\,.$$

On voit aussitôt que la fonction à intégrer est ici positive, ce qui fait l'avantage de la nouvelle formule (7) par rapport aux formules précédentes. On voit directement de (7) que la fonction $F_\alpha(t)$ est positive pour $t > 0$. La formule (7) permet aussi d'obtenir les formules assymptotiques suivantes:

$$F_\alpha(t) \sim Kt^{-(2-\alpha)/(2-2\alpha)}\exp(-At^{-\alpha/(1-\alpha)}) \quad \text{pour} \quad t \to 0\,,$$

et

$$F_\alpha(t) \sim Mt^{-1-\alpha} \quad \text{pour} \quad t \to \infty\,,$$

avec les constantes

$$A = (1-\alpha)\alpha^{\alpha/(1-\alpha)}\,, \quad K = \frac{1}{\sqrt{2\pi(1-\alpha)}}\alpha^{1/(2-2\alpha)}\,, \quad M = \frac{\sin\alpha\pi}{\pi}\Gamma(1+\alpha)$$

(Γ — fonction gamma d'Euler).

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OUVRAGES CITÉS

[1] G. Doetsch, *Handbuch der Laplace-Transformation*, Basel-Stuttgart I, 163.

[2] J. Mikusiński, *Sur les fonctions exponentielles du calcul opératoire*, Studia Math. **12** (1951), 208.

[3] — *Sur la croissance de la fonction opérationnelle* $\exp(-s^\alpha\lambda)$, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 423-425.

[4] H. Pollard, *The representatione of* $e^{-x\lambda}$ *as a Laplace integral*, Bull. Amer. Math. Soc. **52** (1946), 908-910.

[5] A. Wintner, *Cauchy's stable distributions and an explicit formula of Mellin*, Amer. J. Math. **78** (1956), 819-861.

[6] L. Włodarski, *Une remarque sur une classe de fonctions exponentielles du calcul operationnel*, Studia Math. **13** (1953), 188-189.

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MATHEMATICS

On the Isomorphism of the Spaces m and M

by

A. PEŁCZYŃSKI

Presented by S. MAZUR on September 16, 1958

1. We shall use the following standard notations: m – the space of all bounded real sequences $x = (a_n)$ with the norm, $\|x\| = \sup_n |a_n|$,

M – the space of all equivalence classes of bounded measurable *) real functions $x(\ )$ defined on the interval $\langle 0, 1\rangle$ with the norm, $|x| = \operatorname{sup\,ess}_{\langle 0,1\rangle} |x(t)|$,

l – the space of all real sequences $x = (a_n)$ such that $\sum_{n=1}^{\infty} |a_n| < \infty$ with the norm, $\|x\| = \sum_{n=1}^{\infty} |a_n|$,

L – the space of all equivalence classes of measurable real functions $f(\cdot)$ defined on the interval $\langle 0, 1\rangle$ such that $\int_0^1 |f(t)|\,dt < \infty$ with the norm, $\|f\| = \int_0^1 |f(x)|\,dt$,

$\mathfrak{P}_\lambda$ (where λ is a number $\leqslant 1$) – the family of Banach spaces X such that for each Banach space $X' \supset X$ there is a projection (that is a linear idempotent operator) P carrying X' onto X with $\|P\| \leqslant \lambda$.

It is known that two Banach spaces X and Y are called *isomorphic* (we shall write $X \sim Y$), if there is a linear one-to-one function from one space onto the other which preserves topology.

The spaces X and Y have the *same linear dimension*, if the space X is isomorphic to a subspace of Y and Y is isomorphic to a subspace of X.

The space X has a *smaller linear dimension* than Y, if X is isomorphic to a subspace of Y, but not conversely.

---

*) By measurable function we mean the function measurable with respect to Lebesque's measure.

2. THEOREM. *The spaces m and M are isomorphic.*

The proof of this theorem follows from the proposition:

(*) *Let the spaces X and Y satisfy the conditions*:

(i) *for some $\lambda \geqslant 1$ the spaces X and Y belong to $\mathfrak{B}_\lambda$,*

(ii) $X \sim X \times X$ *and* $Y \sim Y \times Y$ *),

(iii) *the spaces X and Y have the same linear dimension.*

Then the spaces X and Y are isomorphic.

Proof of (*). According to (iii) there is a subspace E of Y such that $E \sim X$.

It follows from condition (i) that E is complemented in Y, i. e. there is a subspace $Y_1 \subset Y$ such that $Y = E \times Y_1 \sim X \times Y_1$. Applying the well known properties of the Cartesian product of Banach spaces we obtain from (ii)

$$Y \sim X \times Y_1 \sim (X \times X) \times Y_1 \sim X \times (X \times Y_1) \sim X \times Y .$$

From the symmetry of the conditions (i)-(iii) with respect to the spaces X and Y we also infer that $X \sim X \times Y$. Hence, by reason of the transitivity of the relation of isomorphism, the spaces X and Y are isomorphic.

Proof of the Theorem. It is enough to establish that the spaces m and M satisfy the conditions (i)-(iii). It is known from the results of Akilov [1], Goodner [3] and Nachbin [4] that the spaces m and M belong to $\mathfrak{B}_1$. In [2], 4.2, it has been proved that the spaces m and M have the same linear dimensions. We omit the simple proof that the spaces m and M satisfy condition (ii).

COROLLARY. *The space l has smaller linear dimensions than L, but the conjugate spaces of the spaces l and L are isomorphic.*

This corollary follows immediately from the Theorem because the conjugate spaces of the spaces l and L are isomorphic to spaces m and M, respectively. The fact that l has a smaller linear dimension then L is well known (see e. g. [2], 4.2).

Remark. It is probable that in the Proposition (*) the condition (ii) is extraneous. In particular, the following conjecture is very probable.

Let the spaces X and Y belong to $\mathfrak{B}_1$. Then the spaces X and Y are isomorphic if, and only if, they have the same linear dimension.

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REFERENCES

[1] G. P. Akilov, Dokł. Akad. Nauk. SSSR, **59** (1948), 417.
[2] S. Banach and S. Mazur, Studia Mathematica, **4** (1933), 100.
[3] D. B. Goodner, Trans. Amer. Math. Soc. **69** (1950), 89.
[4] L. Nachbin, Trans. Amer. Math. Soc. **68** (1950), 28.

*) By $E \times F$ we denote the Cartesian product of spaces E and F.

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MATHÉMATIQUE

Étude des systèmes d'équations intégrales singulières pour les arcs non fermés par la méthode des approximations successives

par

D. PRZEWORSKA-ROLEWICZ

Présenté par S. MAZUR le 16 Septembre 1958

Soit l'équation intégrale singulière non linéaire:

$$u(t) = \lambda \int_L \frac{K[t, \tau, u(\tau)]}{\tau - t} d\tau, \tag{1}$$

où L désigne un ensemble de p arcs $L_\nu = a_\nu b_\nu$ *non fermés* et disjoints sur le plan de la variable complexe, $u(t)$ désigne la fonction inconnue, $t, \tau \varepsilon L$. L'intégrale a le sens de la valeur principale de Cauchy.

L'équation (1) a été resolue par A. Goussèinov [1] pour le cas particulier, où L est un segment d'axe réel et la fonction K admet la dérivée K'_u qui vérifie la condition de Lipschitz.

Dans le cas général W. Pogorzelski [2] a étudié les propriétés d'une intégrale singulière et a démontré l'existence au moins d'une solution d'équation (1) pour les valeurs du paramètre λ vérifiant l'inégalité (14′) par l'application du théorème de J. Schauder sur le point invariant d'une transformation, sous la supposition que la fonction K satisfait à la condition de Hölder relativement à t, τ, à la condition de Lipschitz relativement à u, et à l inégalité: $|K(t, \tau, u)| \leqslant k_1|u|^r + k_2$ $(0 < r < 1)$.

Dans ce travail nous démontrerons, qu'il est possible d'appliquer la méthode des approximations successives à des systèmes finis ou infinis des équations de la forme (1). Autrement dit, que dans le cas des équations intégrales ordinaires la méthode des approximations successives pour les équations singulières entraîne des difficultés et n'était appliquée jusqu'à présent que dans le cas particulier de Goussèinov.

Introduction

Soit L un ensemble de p arcs non fermés dirigés $a_\nu b_\nu$ $(\nu = 1, 2, \ldots p)$ sur le plan de la variable complexe, qui ont la tangente continue en tout point et qui sont disjoints.

Nous désignons

$$P_{t,\tau} = \prod_{\nu=1}^{p} |t-a_\nu| \cdot |\tau - b_\nu| \tag{2}$$

en admettant que les points t, τ sont situés sur le même arc $a_\nu b_\nu$, et τ suit t dans la direction positive de l'arc $a_\nu b_\nu$.

Soit un espace de Banach X. Nous désignons par C_X^α l'espace des fonctions continues $u(t)$ définies pour $t \in L$, qui prennent des valeurs appartenant à l'espace X et qui vérifient l'inégalité:

$$\|u(t)\|_X \leqslant c_u P_{t,t}^{-\alpha},$$

c_u étant une constante positive arbitraire, α — une constante positive inférieure à l'unité, fixée pour tout espace C_X^α. Dans l'espace C_X^α on peut définir l'intégrale de la fonction $u(t)$ par la limite suivante

$$\int_L u(t)\,dt = \sum_{\nu=1}^{p} \lim_{t_{1\nu}\to a_\nu} \lim_{t_{2\nu}\to b_\nu} \int_{t_{1\nu}}^{t_{2\nu}} u(t)\,dt,$$

où les points $t_{1\nu}, t_{2\nu}$ sont situés à l'intérieur de l'arc $a_\nu b_\nu$. On voit que toute la fonction $u(t) \in C_X^\alpha$ est intégrable dans ce sens.

S'il existe la limite suivante

$$\lim_{\varepsilon\to 0} \int_{L-l_\varepsilon} \frac{u(\tau)}{\tau - t}\,d\tau,$$

où l_ε désigne une portion des arcs $a_\nu b_\nu$ située à l'intérieur du cercle de centre t et du rayon ε, nous la désignerons par $\int_L \frac{u(\tau)}{\tau - t}\,d\tau$ et nous l'appellerons *intégrale singulière au sens de Cauchy.*

Soit l'espace H_μ^α des fonctions continues de la variable $t \in L$, qui ont les valeurs de l'espace X et qui vérifient les inégalités:

$$\|u(t)\|_X \leqslant c' P_{t,t}^{-\alpha}, \tag{3}$$

$$\|u(t)-u(t_1)\|_X \leqslant c P_{t,t_1}^{-\alpha-\mu} |t-t_1|^\mu, \tag{4}$$

$$c', c > 0; \quad \alpha, \mu > 0; \quad \alpha+\mu < 1. \tag{4'}$$

Nous désignons:

$$s_\mu(u) = \sup_{t\in L} P_{t,t} \|u(t)\|_X;$$

$$h_u^\alpha(u) = \sup_\nu \sup_{t,t_1\in L_\nu} P_{t,t_1}^{\alpha+\mu} \frac{\|u(t)-u(t_1)\|_X}{|t-t_1|^\mu}.$$

L'espace H^{α}_{μ} est l'espace de Banach avec la norme

$$\|u\|_{H^{\alpha}_{\mu}} = s_{\mu}(u) + h^{\alpha}_{\mu}(u) \tag{5}$$

(les exposants α, μ sont fixés pour tout espace H^{α}_{μ}).

Soit l'espace $H^{\alpha}_{\mu,\nu}$ des fonctions continues $u(t, \tau)$ des variables $t, \tau \in L$, qui ont les valeurs de X et qui vérifient les inégalités:

$$\|u(t, \tau)\|_X \leqslant c' P^{-\beta}_{\tau,\tau} , \tag{6}$$

$$\|u(t, \tau) - u(t_1, \tau_1)\|_X \leqslant c P^{-\alpha-\mu}_{\tau,\tau_1} [|t-t_1|^{\nu} + |\tau-\tau_1|^{\mu}] , \tag{7}$$

$$c', c > 0; \quad 0 < \mu < \nu \leqslant 1; \quad 0 < \beta < \alpha < 1-\mu . \tag{7'}$$

Nous désignons:

$$s_{\mu,\nu}(u) = \sup_{t,\tau \in L} P^{\beta}_{\tau,\tau} \|u(t, \tau)\|_X ;$$

$$h^{\alpha}_{\mu,\nu}(u) = \sup_{\nu} \sup_{t,t_1,\tau,\tau_1 \in L_{\nu}} P^{\alpha+\mu}_{\tau,\tau_1} \frac{\|u(t, \tau) - u(t_1, \tau_1)\|_X}{[|t-t_1|^{\nu} + |\tau-\tau_1|^{\mu}]} .$$

L'espace $H^{\alpha}_{\alpha,\nu}$ est l'espace de Banach avec la norme

$$\|u\|_{H^{\alpha}_{\mu,\nu}} = s_{\mu,\nu}(u) + h^{\alpha}_{\mu,\nu}(u) . \tag{8}$$

Nous présenterons maintenant les deux théorèmes fondamentaux, concernant les propriétés d'une intégrale de Cauchy, démontrés par W. Pogorzelski [2] dans le cas où X désigne tout le plan de la variable complexe. Leurs démonstrations sont valables sans modifications essentielles pour l'espace arbitraire X de Banach.

THÉORÈME 1. *Si $u(t, \tau) \in H^{\alpha}_{\mu,\nu}$, alors la fonction*

$$\hat{u}(t) = \int_L \frac{u(t, \tau)}{\tau - t} d\tau \tag{9}$$

appartient à l'espace H^{α}_{μ} et $s_{\mu}(\hat{u}) \leqslant C'_1 s_{\mu,\nu}(u) + C_1 h^{\alpha}_{\mu,\nu}(u)$, où les constantes positives C_1, C'_1 ne dépendent que des lignes L.

THÉORÈME 2. *Si $u(t, \tau) \in H^{\alpha}_{\mu,\nu}$ alors $\hat{u}(t) \in H^{\alpha}_{\mu}$ et $h^{\alpha}_{\mu}(\hat{u}) \leqslant C'_2 s_{\mu,\nu}(u) + C_2 h^{\alpha}_{\mu,\nu}(u)$, où les constantes positives ne dépendent que des lignes L.*

Il en résulte immédiatement le corollaire suivant.

COROLLAIRE 1. *Si $u(t, \tau) \in H^{\alpha}_{\mu,\nu}$, alors $\hat{u}(t) \in H^{\alpha}_{\mu}$ et $\|\hat{u}\|_{H^{\alpha}_{\mu}} \leqslant C \|u\|_{H^{\alpha}_{\mu,\nu}}$, où la constante positive ne dépend que des lignes L.*

Solution du problème

Soit l'ensemble Z défini par les inégalités:

$$s_{\mu}[u(t)] \leqslant \varrho; \quad h^{\alpha}_{\mu}[u(t)] \leqslant \varkappa , \tag{10}$$

où $u(t) \in C^{\alpha}_X$, ϱ et $\varkappa$ sont des constantes positives arbitrairement fixées.

On voit que $u \in H^{\alpha}_{\mu}$, si $u \in Z$.

Soit l'opération $K(t, \tau, \xi)$, déterminée sur l'ensemble $\mathcal{L}[t, \tau \in L;\ \xi \in X]$ qui prend les valeurs appartenant à l'espace X et qui vérifie les inégalités:

$$\|K(t, \tau, \xi)\|_X \leqslant k'\|\xi\|^r + k'', \tag{11}$$

$$\|K(t, \tau, \xi) - K(t_1, \tau_1, \xi_1)\|_X \leqslant k[|t-t_1|^{\nu} + |\tau-\tau_1|^{\mu} + \|\xi-\xi_1\|_X], \tag{11'}$$

$$\left(k,\ k',\ k'' > 0; \quad 0 < \mu < \nu \leqslant 1; \quad 0 < r \leqslant \frac{\beta}{\alpha} \leqslant 1\right).$$

Théorème auxiliaire 1. *Si* $u(t) \in Z$, *alors* $Ku = K[t, \tau, u(\tau)] \in H^{\alpha}_{\mu,\nu}$ *et les inégalités suivantes sont satisfaites*:

$$s_{\mu,\nu}(Ku) \leqslant k' m_2 \varrho^r + k''; \quad h^{\alpha}_{\mu,\nu}(Ku) \leqslant k(m_1 + \varkappa), \tag{12}$$

où l'on a désigné

$$m_1 = \max[1, \sup_{\nu} \sup_{t,t_1 \in L_{\nu}} P^{\alpha+\mu}_{t,t_1}]; \quad m_2 = \sup_{\nu} \sup_{t,\tau \in L_{\nu}} P^{\beta-\alpha r}_{t,\tau}$$

(les constantes positives m_1, m_2 *ne dépendent que des lignes* L*).*

Démonstration. Elle résulte immédiatement des inégalités (10), (11) et (11').

On peut traiter l'opération Ku comme une opération qui transforme l'espace H^{α}_{μ} dans l'espace $H^{\alpha}_{\mu,\nu}$.

Soit l'opération intégrale singulière suivante:

$$\lambda \hat{K}u = \lambda \int_L \frac{K[t, \tau, u(\tau)]}{\tau - t} d\tau. \tag{13}$$

Il résulte du théorème auxiliaire 1 et du corollaire 1, que l'opération $\lambda \hat{K}u$ transforme l'ensemble Z en un certain ensemble $\lambda \hat{K} Z \subset H^{\alpha}_{\mu}$. Nous démontrerons le théorème suivant:

Théorème 3. *Si le paramètre* λ *verifie l'inégalité suivante*:

$$|\lambda| < \min\left[\frac{\varrho}{C'_1(k' m_2 \varrho^r + k'') + C_1 k(m_1 + \varkappa)};\ \frac{\varkappa}{C'_2(k' m_2 \varrho^r + k'') + C_2 k(m_1 + \varkappa)}\right], \tag{14}$$

alors $\lambda \hat{K} Z \subset Z$.

Démonstration. Il résulte du théorème auxiliaire 1 et des théorèmes 1 et 2 que les inégalités suivantes sont satisfaites:

$$s_{\mu}(\hat{K}u) \leqslant C'_1 s_{\mu,\nu}(Ku) + C_1 h^{\alpha}_{\mu,\nu}(Ku) \leqslant C'_1(k' m_1 \varrho^r + k'') + C_1 k(m_1 + \varkappa),$$

$$h^{\alpha}_{\mu}(\hat{K}u) \leqslant C'_2(k' m_2 \varrho^r + k'') + C_2 k(m_1 + \varkappa),$$

d'où résultent les inégalités suffisantes pour que l'ensemble $\lambda\hat{K}Z$ fasse partie de l'ensemble Z:

$$C_2'(k'm_2\varrho^r + k'') + C_1 k(m_1 + \varkappa) \leqslant \varrho;$$

$$C_2'(k'm_2\varrho^r + k'') + C_2 k(m_1 + \varkappa) \leqslant \varkappa$$

et nous en déduirons l'inégalité (14).

Remarque. Les constantes positives ϱ, $\varkappa$ étant arbitraires nous pouvons les choisir d'une telle maniére, que l'intervalle pour le paramètre λ soit le plus grand possible. Nous obtenons (de la même façon que dans le travail [2]) la limitation suivante:

$$|\lambda| < \frac{1}{C_2 k}, \tag{14'}$$

d'où résulte le corollaire suivant:

Corollaire 2. *Si le paramètre λ vérifie l'inégalité* (14'), *alors* $\lambda\hat{K}Z \subset Z$.

Supposons encore que l'opération Ku, déterminée sur l'ensemble $\mathcal{L}$ et vérifiant les conditions (11) et (11'), satisfait à la condition suivante, dite condition (W^a):

$$s_{\mu,\nu}(Ku_1 - Ku_2) \leqslant d_1 s_\mu(u_1 - u_2) + d_2 h_\mu^a(u_1 - u_2), \tag{15}$$

$$h_{\mu,\nu}^a(Ku_1 - Ku_2) \leqslant d_3 s(u_1 - u_2) + d_4 h_\mu^a(u_1 - u_2), \tag{15'}$$

si $u_1, u_2 \in H_\mu^a$ (les constantes positives d_1, d_2, d_3, d_4 ne disparaisent pas simultanément).

Nous remarquons que la classe des opérations satisfaisant à la condition (W^a) est linéaire. En outre, si les opérations Ku et $\tilde{K}u$ satisfont à cette condition avec les constantes (d_i) resp. $(\tilde{d}_i)$ $(i = 1, 2, 3, 4)$, alors l'opération $Ku + \tilde{K}u$ satisfait à la condition (W^a) avec les constantes $(d_i + \tilde{d}_i)$.

Théorème 4. *Si l'opération Ku satisfait à la condition* (W^a), *alors l'inégalité suivante est satisfaite*:

$$\|\lambda\hat{K}u_1 - \lambda\hat{K}u_2\|_{H_\mu^a} \leqslant |\lambda| q \|u_1 - u_2\|_{H_\mu^a}, \tag{16}$$

où

$$q = \max[(C_1 + C_2')d_1 + (C_1 + C_2)d_3;\ (C_1' + C_2')d_2 + (C_1 + C_2)d_4]. \tag{17}$$

Démonstration. Nous avons

$$\hat{K}u_1 - \hat{K}u_2 = \lambda \int_L \frac{K[t, \tau, u_1(\tau)] - K[t, \tau, u_2(\tau)]}{\tau - t} d\tau,$$

d'où, en nous appuyant sur les théorèmes 1, 2, nous obtiendrons:

$$s_\mu(\lambda\hat{K}u_1 - \lambda\hat{K}u_2) \leqslant |\lambda| \{(C_1' d_1 + C_1 d_3) s_\mu(u_1 - u_2) + (C_1' d_2 + C_1 d_4) h_\mu^a(u_1 - u_2)\},$$

$$h_\mu^a(\lambda\hat{K}u_1 - \lambda\hat{K}u_2) \leqslant |\lambda| \{(C_2' d_1 + C_2 d_3) s_\mu(u_1 - u_2) + (C_2' d_2 + C_2 d_4) h_\mu^a(u_1 - u_2)\}.$$

Il en résulte l'inégalité proposée:

$$\|\lambda\hat{K}u_1-\lambda\hat{K}u_2\|_{H_\mu^a}\leqslant|\lambda|\{[(C_1'+C_2')d_1+(C_1+C_2)d_3]s_\mu(u_1-u_2)+$$
$$+[(C_1'+C_2')d_2+(C_1+C_2)d_4]h_\mu^a(u_1-u_2)\}\leqslant|\lambda|q\|u_1-u_2\|_{H_\mu^a}.$$

Théorème 5. *Si l'opération $K(t,\tau,\xi)$, déterminée sur l'ensemble $\mathcal{L}$, vérifie les inégalités* (11) *et* (11′) *et satisfait à la condition (W^a), alors pour les valeurs du paramètre λ suffisamment petites, notamment*

$$|\lambda|\min<\left[\frac{1}{q};\frac{1}{C_2k}\right], \tag{18}$$

l'équation intégrale singulière

$$u(t)=\lambda\hat{K}u \tag{19}$$

a une solution unique, qui appartient à l'ensemble Z.

La démonstration résulte immédiatement du corollaire 2, du théorème 5 et du théorème de Banach sur le point invariant d'une transformation.

La classe $\mathcal{H}_{\mu,\nu}$ des opérations

Passons à la détermination de la classe la plus étendue des opérations satisfaisantes à la condition (W^a).

Définition. L'opération $G(t,\tau,\xi)$ déterminée sur l'ensemble $\mathcal{L}$, qui a les valeurs de l'espace X, appartient à la classe $\mathcal{H}_{\mu,\nu}$, si elle satisfait à la condition suivante:

$$G(t,\tau,\xi_1)-G(t,\tau,\xi_2)=\mathfrak{G}_1(t,\tau,\xi_1,\xi_2)[\mathfrak{G}_2(\xi_1-\xi_2)], \tag{20}$$

où 1) l'opération $\mathfrak{G}_2(\xi)$ transforme l'espace X en lui-même; 2) $\mathfrak{G}_2(0)=0$; 3) $\mathfrak{G}_2$ satisfait à la condition de Lipschitz:

$$\|\mathfrak{G}_2(\xi)-\mathfrak{G}_2(\xi')\|_X\leqslant\mathfrak{g}_2\|\xi-\xi'\|_X; \tag{21}$$

4) l'opération $\mathfrak{G}_1(t,\tau,\xi_1,\xi_2)[\xi]$ est linéaire pour t,τ,ξ_1,ξ_2 fixés et transforme l'espace X en lui-même; 5) $\mathfrak{G}_1$ vérifie les inégalités

$$\|\mathfrak{G}_1(t,\tau,\xi_1,\xi_2)\|_{X\to X}\leqslant\mathfrak{g}_1'P_{t,\tau}^{\alpha-\beta}, \tag{22}$$

$$\|\mathfrak{G}_1(t,\tau,\xi_1,\xi_2)-\mathfrak{G}_1(t',\tau',\xi_1',\xi_2')\|_{X\to X}\leqslant\mathfrak{g}_1[|t-t'|^\nu+ \tag{22′}$$
$$+|\tau-\tau'|^\mu+\|\xi_1-\xi_1'\|_X+\|\xi_2-\xi_2'\|_X];$$
$$(\mathfrak{g}_1,\mathfrak{g}_1',\mathfrak{g}_2>0;\ 0<\mu<\nu\leqslant1;\ 0<\beta\leqslant\alpha\leqslant1-\mu),$$

où $X\to X$ désigne l'espace des opération linéaires qui transforment l'espace X en lui-même.

Il résulte des suppositions 2) et 3) que $\|\mathfrak{G}_2(\xi)\|_X\leqslant\mathfrak{g}_2\|\xi\|_X$.

THÉORÈME 6. *Si l'opération* $G(t, \tau, \xi) \in \mathcal{H}_{\mu,\nu}$, *alors elle satisfait à la condition* (W^a) *avec les constantes suivantes*:

$$d_1 = g_1' g_2; \quad d_2 = 0; \quad d_3 = g_1 g_2 m_3; \quad d_4 = m_4 d_1, \tag{24}$$

où l'on a designé

$$m_3 = \sup_{\nu} \sup_{\tau,\tau_1 \in L_\nu} P^{\mu}_{\tau,\tau_1}(1+2\varkappa); \quad m_4 = \sup_{\nu} \sup_{t,\tau \in L_\nu} P^{\alpha-\beta}_{t,\tau}.$$

Démonstration. Elle résulte facilement des suppositions concernant les opérations de la classe $\mathcal{H}_{\mu,\nu}$.

Remarque. Il est intéressant, que les opérations de la classe $\mathcal{H}_{\mu,\nu}$ (sans la supposition (22)) nous permettent de résoudre l'équation (19) pour les arcs fermés par la méthode des approximations successives, malgré les propriétés essentiellement différentes de l'intégrale et des opérations Ku et $\hat{K}u$ [3]. En outre, on peut démontrer, que chaque opération ayant la différentielle forte (satisfaisant à la condition de Lipschitz) est de la classe $\mathcal{H}_{\mu,\nu}$. Au contraire, sous une certaine supposition concernant l'opération $\mathfrak{G}_2$, chaque opération de la classe $\mathcal{H}_{\mu,\nu}$ admet la différentielle forte, qui satisfait à la condition de Lipschitz.

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OUVRAGES CITÉS

[1] A. I. Goussèinov, *Sur une équation intégrale* (en russe). Izviestia AN SSSR **12** (1948), 193.

[2] W. Pogorzelski, *Propriétés d'une intégrale singulière pour les arcs non fermés et leur application*. Bull. Acad. Polon. Sci., Serie de sci. math., astr. et phys. **6** (1958), 85.

[3] D. Przeworska-Rolewicz, *Sur l'application de la méthode des approximations successives à l'équation intégrale à forte singularité*. Ann. Pol. Math. (sous presse).

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EXPERIMENTAL PHYSICS

The Influence of Torsional Vibrations of Luminescent Molecules on the Fundamental Polarization of Fluorescence

by

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Presented by A. JABŁOŃSKI on July 12, 1958

The term "fundamental polarization" denotes the degree of polarization of the photoluminescence of an isotropic solution in the absence of any depolarizing factors.

Three theories of the fundamental polarization of photoluminescence exist at present. The first of these was given by Vavilov and Levshin in 1923 [1], the second by F. Perrin in 1925 [2], and the third by A. Jabłoński in 1935 [3]-[5] and 1936 [6]. Of these, Jabłoński's theory presents the highest degree of generality, and yields values of the fundamental polarization P within the following limits:

$$-\tfrac{1}{3} \leqslant P \leqslant \tfrac{1}{2}\,. \tag{1}$$

Actually, the values of the fundamental polarization obtained experimentally are restricted to

$$-\tfrac{1}{3} < P < \tfrac{1}{2}\,. \tag{2}$$

It has never been possible to achieve the limiting values as given by (1), even after all presumable depolarizing factors had been removed.

A. Jabłoński [7] assumed this to be due to the existence of certain additional depolarizing factors, hitherto not accounted for, the main one being the torsional vibrations of the luminescent molecules about their orientations of equilibrium. The energy of the torsional vibrations depends on the temperature. From the theoretical considerations of A. Jabłoński [8] it appears that the fundamental polarization depends on the temperature *via* the dependence on the torsional vibrations.

The aim of the present paper was to investigate the temperature dependence of the fundamental polarization.

The investigation of this dependence was found to present considerable difficulties due to the fact that the theory [9] predicted an

effect of the temperature on fundamental polarization of the order of accuracy of the best known and mostly employed experimental method (the Wood-Dunoyer visual compensation method).

This explains why the earlier investigations failed to establish the temperature dependence of the fundamental polarization. Investigations on the subject were carried by J. Cahen [10], A. Jabłoński [11], P. Pesteil and M. Barbaron [12], and E. Laffitte [13].

The present author has reduced the mean error of the method by performing 12-25 measurements at each of the various temperatures, thus establishing the existence of the effect under consideration, comparable in the magnitude to that of the accuracy of the method applied.

Experimental

Measurements were made with rigid solutions of anthracene in methyl metacrylate, at concentrations of 10^{-5} g/cm³ to 10^{-3} g/cm³. The solutions were prepared by the method described in [14].

The degree of polarization of the samples was measured at room temperature and at a temperature of about −150°C obtained in a Dewar flask filled with liquid air.

The flask was of very simple construction (Fig. 1), and consisted of two parts separated by an insulated screen *A*, of the liquid air tank *B*, and of the head *C* with four plexiglass windows. The luminophor was placed upon the upper end of the copper rod 1, the lower end of which was immersed in liquid air. The tubes in the openings 1, 2, 3 in the screen *A* were tightened with vacuum wax and shellac, to prevent the passing of liquid air from *B* to *C*. The tubes 2 and 3 serving to introduce the liquid air and to let out its vapours were also sealed near their outlet to the head *C*. After placing a new sample within *C*, its contact surface with the partition plate *A* was covered with a layer of shellac. Thus, the air contained within *C* was entirely cut off from the outer atmosphere and could be dried by the hygroscopic substance (fine silicagel crystals) placed within the head *C*.

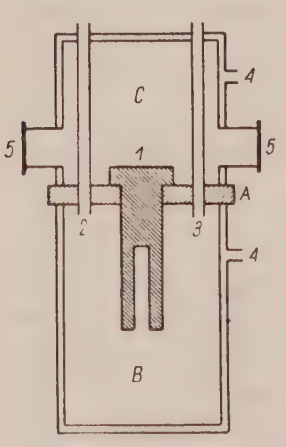

Fig. 1. Dewar flask as employed in investigation: 1 — copper rod for cooling sample, 2, 3 — tubes for introducing liquid air and removing its vapours 4 — to vacuum pumps, 5 — plexiglass windows. For other explanations, see text

This Dewar flask presented the advantage of making all manipulations easy and not requiring the luminophor to be placed *in vacuo* for protection from rime.

The temperature was measured with a thermocouple.

Photoluminescence of the anthracene was excited with light from a HQE 40 mercury lamp filtered by a Wood filter. Samples of high concentrations of the dye were irradiated with both polarized and natural light, whilst those of low concentrations were excited with natural light only, since the intensity of the fluorescence of samples excited with polarized light was so low as to render measuring the degree of polarization impossible. Degrees of polarization P_n obtained by excitation with natural light were converted into degrees of polarization P_p by excitation with polarized light by means of Vavilov's relation [15]:

$$P_p = \frac{2P_n}{1+P_n}. \tag{3}$$

The degree of polarization was measured by the visual compensation method, using a Savart plate and Arago compensator. The fluorescence light was observed perpendicularly to the direction of the exciting beam. A schematic representation of the experimental device has been given in an earlier paper [14].

The degree of polarization was computed from the Kawski-Neumann formula [16] for an Arago compensator with four plates:

$$P = \frac{\left(1-\frac{1}{n^2}\right)^2 \cdot \sin^2\alpha}{\left(1+\frac{1}{n^2}\right)\left[2-\left(1+\frac{1}{n^2}\right)\sin^2\alpha\right] - \frac{3\cos\alpha}{n^2}\sqrt{n^2-\sin^2\alpha}}, \tag{4}$$

where n denotes the refractive index of plates, and α – the angle at which the fringes disappear.

Results

The angle at which the interference fringes disappear in the measuring system (Savart plate + analyzer) was measured several times for each sample at room temperature and at about −150°C.

The fiducial limits of the measurements were determined by the theory of errors assuming a probability level of 5 per cent. The fiducial limits were determined as follows.

Let us denote by

$$t = \frac{\varphi-\psi}{s/\sqrt{n}}, \tag{5}$$

where φ is the mean arithmetical value of n measurements,
ψ – the true value of the physical quantity,
n – the number of measurements, and
$s = \sqrt{\frac{\Sigma(\varphi_i-\varphi)^2}{n-1}}$ (φ_i being the value obtained in the i-th measurement).

The probability 95% that the inequality $\left|\frac{\varphi-\psi}{s/\sqrt{n}}\right| \leqslant t(5\%, n)$ is fulfilled

$$P\left(\left|\frac{\varphi-\psi}{s/\sqrt{n}}\right| \leqslant t\ (5\%, n)\right) = 95\%$$

may by calculated by using the tabulated values of $t(p, n)$ (where p is the probability level). By suitably transforming Eq. (5) the interval of reliability containing the true value of the degree of polarization is found:

$$\left[\varphi - \frac{s}{\sqrt{n}}\, t(5\%, n),\quad \varphi + \frac{s}{\sqrt{n}}\, t(5\%, n)\right].$$

Table I gives the values of $t(5\%, n)$ quoted from Fisher and Yates' Tables [19].

TABLE I

Values of quantity $t(5\%, n)$, at 5% probability level, n — number of measurements

n	3	12	14	15	16	18	25
$t(5\%, n)$	4.303	2.201	2.16	2.145	2.131	2.11	2.0

The results of the measurements are given in Table II and in Figs. 2 and 3. In Fig. 2, the ordinates represent the degree of polarization P in per cents, whilst the abscissae express the concentration c in g/cm³. In Fig. 3, the co-ordinates represent respectively, $\frac{1}{P_p}$, and c in g/cm³. The curves are for temperatures $T_1 = 295°\mathrm{K}$ and $T_2 = 125°\mathrm{K}$.

The shape of the curves in Fig. 2 is in agreement with A. Jabłoński's theory of concentrational depolarization [17]. The curve in Fig. 2 for $T_1 = 295°\mathrm{K}$ is in good agreement with that obtained by A. Kawski for anthracene [18].

The points of intersection of the straight lines with the ordinate for $c = 0$ in Fig. 3 yield the following values of the fundamental polarization $P'_p(T)$:

$$\text{for } T_1 = 295°\mathrm{K} \qquad P'_p = 26.6\%$$
$$\text{for } T_2 = 125°\mathrm{K} \qquad P'_p = 27.8\%.$$

Plotting P'_p (per cent) *versus* T (Fig. 4) the fundamental polarization P for $T = 0°\mathrm{K}$ may be found to be (approximately) $P = 28.7\%$ *).

*) According to the terminology due to A. Jabłoński, the quantity P in Eq. (6) is termed fundamental polarization, whilst P' is termed pseudo-fundamental polarization. Throughout the present paper, for brevity, the term fundamental polarization is employed, although it is only the pseudo-fundamental polarization that is observed experimentally.

TABLE II

Rate of polarization of fluorescence of anthracene in plexiglass, at temperatures $T_1 = 295°$ and $T_2 = 125°\,K$

	$T_1 = 295°\,K$					$T_2 = 125°\,K$				
c	10^{-5}	$5\cdot10^{-5}$	$8\cdot10^{-5}$	$1.9\cdot10^{-4}$	$8.7\cdot10^{-4}$	10^{-5}	$5\cdot10^{-5}$	$8\cdot10^{-5}$	$1.9\cdot10^{-4}$	$8.7\cdot10^{-4}$
n	18	16	16	12	15	15	15	3	14	25
φ	35.1	34	33.3	32.4	38	36.1	34.9	34.2	33.4	39.2
P_p^-	25.7	24.5	23.9	22.5	17.9	27.1	25.4		23.6	19.1
P_p^+	26.7	25.3	24.5	23.5	18.5	28.1	26.6		24.6	19.7
P_p	26.2	24.9	24.2	23	18.2	27.5	26	25.2	24.1	19.4
$1/P_p$	0.0381	0.0401	0.0413	0.0434	0.0543	0.0363	0.0385	0.0397	0.0415	0.0525

Explanation of symbols:

P_p^-, P_p^+ — limit values of rate of polarization as obtained from fiducial limits

P_p — rate of polarization in per cent

c — concentration in g/cm³

n — number of measurements

φ — angle at which interference fringes disappear, in degrees

Introducing the respective values of $P'_p(T_1)$, $P'_p(T_2)$ and P in Jabłoński's formula [9]:

$$P'_p = P\left[1-(6-2P)\frac{\hbar}{I\omega}\left(\frac{1}{2}+\frac{KT}{\hbar\omega}\right)\right], \tag{6}$$

and solving the system of two equations (6) for T_1 and T_2, it is possible to calculate the moment of inertia I and mean angular frequency of tor-

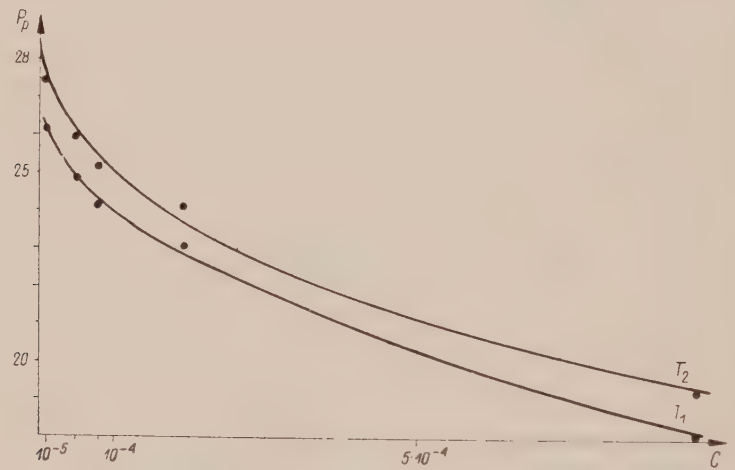

Fig. 2. P_p in per cents *versus* c in g/cm³. Curves T_1 and T_2 are for temperatures 295° and 125° K, respectively

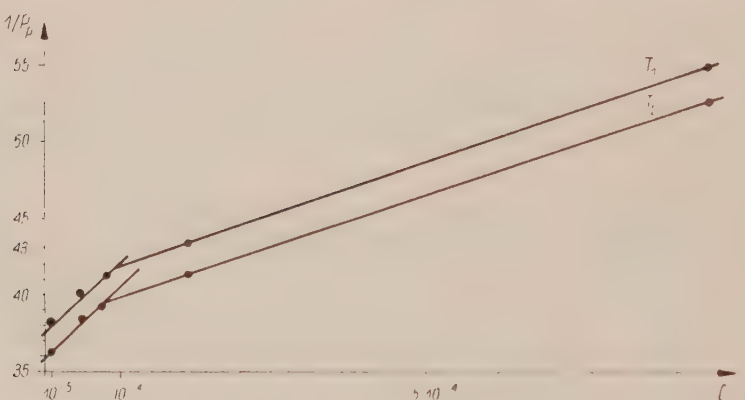

Fig. 3. $1/P_p \cdot 10^3$ *versus* c in g/cm³. Curves as in Fig. 1

sional vibrations ω for anthracene. Calculations yield $\omega = 4.4 \times 10^{12}$ sec^{-1} (wave number 23 cm^{-1}), and $I = 15.3 \times 10^{-38}$ g/cm².

The moment of inertia was also calculated from the structural formula of anthracene, assuming a distance of 1.41 Å between the carbon atoms as given by Houben-Weyl [20]. The value obtained was $I_s = 17.9 \times$

$\times 10^{-38}$ g/cm^2, which, when introduced into expression

$$\omega' = \sqrt{\frac{k(T_1 - T_2)}{(\delta_a - \delta_b) I_s}}, \tag{7}$$

resulting from transformation of the system of two equations of type (6), with k denoting Boltzmann's constant, and

$$\delta_a = \delta(T_1) = \frac{P - P'_p(T_1)}{2P(3-P)}, \qquad \delta_b = \delta(T)_2 \frac{P - P'_p(T_2)}{2P(3-P)},$$

yielded a frequency of torsional vibrations amounting to $\omega' = 4.1 \times \times 10^{12}$ sec^{-1} (wave number 22 cm^{-1}). It will be noted that the frequencies calculated by the two different methods are in excellent agreement.

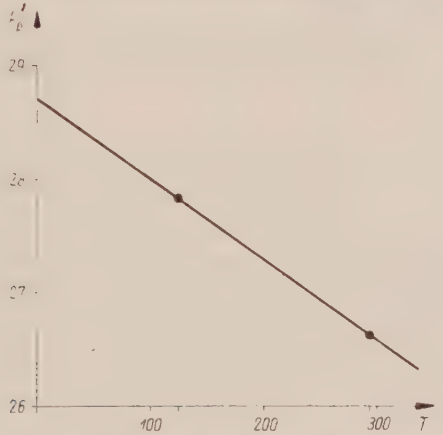

Fig. 4. P'_p versus temperature T°K. At $T = 0°$ K, fundamental polarization $P = 28.7\%$.

The values of ω and I obtained for anthracene seem to be reasonable, when compared to the known values for benzene.

The author wishes to express his thanks to Professor A. Jabłoński for suggesting the subject and for his highly valuable advice, and is indebted to his co-worker M. Czajkowski for his help in making the measurements.

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REFERENCES

[1] S. I. Vavilov, V. L. Levshin, Zt. f. Phys. **16** (1923), 135.
[2] F. Perrin, C. r. Acad. Sci., Paris **180** (1925), 581.
[3] A. Jabłoński, Zt. f. Phys. **96** (1935a), 236.
[4] — Acta Phys. Polon. **4** (1935b), 371, 384.
[5] — Phys. Zt. Sow. Union, **8** (1935c), 105.
[6] — Acta Phys. Polon. **5** (1936), 271.
[7] — Acta Phys. Polon. **10** (1950), 33.

[8] — Acta Phys. Polon. **10** (1950), 193.
[9] — Il N. Cimento ser. X, **2** (1955), 995.
[10] J. Cahen, Journal de chim. phys., **30** (1933), 420.
[11] A. Jabłoński, Acta Phys. Polon. **7** (1938), 15.
[12] P. Pesteil and M. Barbaron, Journ. de Phys. et de Rad., **237** (1953), 884.
[13] E. Laffitte, Ann. de phys., **10** (1955), 71.
[14] M. Czajkowski, J. Grzywacz, Bull. Acad. Polon. Sci., Série des sci. math., astr. et phys. **6** (1958), 107.
[15] S. I. Vavilov, Zt. f. Phys., **16** (1923), 135.
[16] A. Kawski, Optik **14** (1957), 260.
[17] A. Jabłoński, Acta Phys. Polon., **13** (1954), 175; **14** (1955), 295.
[18] A. Kawski, Bull. Acad. Polon. Sci., Série des sci. math., astr. et phys. (in press).
[19] Fisher and Yates, *Statistical Tables*. Table III (1938), Edinburgh.
[20] Houben-Weyl, *Methoden der Org. Chemie*, vol. III, Part 2, p. 572, Stuttgart (1955), Georg Thieme Verlag.

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EXPERIMENTAL PHYSICS

On the Temperature Dependence of the Fundamental Polarisation

by

C. BOJARSKI

Presented by A. JABŁOŃSKI on August 19, 1958

It is known that the experimental values of the fundamental polarization *) of fluorescence of luminescent molecules in rigid solutions or in solutions exhibiting a very large coefficient of viscosity are always lower than those prediceted by theory [1], [2]. To give an explanation of this fact, A. Jabłoński [3] assumed that, in addition to Brown's rotational motion and migration of the excitation energy, other depolarizing factors exist, the chief ones are torsional vibrations of the luminescent molecules about their equilibrium orientation. The theory of fundamental polarization proposed by A. Jabłoński [4], [5] yields the following expression for the degree of principal polarization **) for molecules at absolute rest:

$$P_p = \frac{3\sum_{i=1}^{3} \Gamma_i^2 - \left(\sum_{j=1}^{3} \Gamma_j\right)^2}{\sum_{i=1}^{3} \Gamma_i^2 + 3\left(\sum_{j=1}^{3} \Gamma_j\right)^2}, \tag{1}$$

where Γ_j denotes the relative principal transition probabilities along the three mutually perpendicular axes of the virtual electronic oscillator. The theory, as subsequently generalized by A. Jabłoński [3] for luminescent molecules executing torsional vibrations, leads to the following

*) i. e. the limit value of the polarization of the photoluminescence of an isotropic solution in which depolarizing factors, such as Brown's rotational motion of the luminescent molecules and migration of the excitation energy from one molecule to another, are absent.

**) if one and the same virtual electronic oscillator accounts for both the absorption and the emission of light, the fundamental polarization is termed principal polarization.

expression for the pseudoprincipal polarization *) in the case of a linear oscillator:

$$P'_p = \frac{9u^2 - 12u + 4}{3u^2 - 4u + 8} \tag{2}$$

and, in that of a symmetrical plane oscillator:

$$P'_p = \frac{9u^2 - 12u + 4}{3u^2 - 4u + 28}, \tag{3}$$

where the quantity

$$\frac{u}{2} = \bar{\varepsilon}^2 = \frac{\hbar}{I \cdot \omega}\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1}\right) \tag{4}$$

is the mean square angle between the axis of the luminescent molecule and its equilibrium orientation, with I denoting the moment of inertia of the molecule and ω — the angular frequency of its torsional vibrations about the corresponding axis of inertia.

On introducing certain simplifications, A. Jabłoński [6] obtained the following approximate expression for the temperature dependence of the degree of pseudoprincipal polarization:

$$P'_p = P\left[1 - (6 - 2P_p)\frac{\hbar}{I \cdot \omega}\left(\frac{1}{2} + \frac{kT}{\hbar\omega}\right)\right]; \tag{5}$$

here, P'_p and P_p denote the pseudoprincipal and principal polarizations, respectively. The results of A. Jabłoński's theory, as given by Eqs. (2) and (3), may be proved to yield the temperature dependence of the pseudoprincipal polarization in a shape more general than that of Eq. (5). By simple transformations, Eqs. (2) and (3) assume the form

$$\frac{1}{P'_p} = \frac{1}{3} + \frac{A}{(\frac{3}{2}u - 1)^2}, \tag{6}$$

where A is a constant assuming the values of $\frac{2}{3}$ and $\frac{2}{3}^0$ for the linear and plane oscillator, respectively. For $u = 0$ (i. e., in the absence of torsional vibrations), Eq. (6) yields

$$\frac{1}{P_p} = \frac{1}{3} + A. \tag{7}$$

By (7), Eq. (6) may be rewritten as follows:

$$\frac{1}{P'_p} - \frac{1}{3} = \left(\frac{1}{P_p} - \frac{1}{3}\right) \cdot \frac{1}{(\frac{3}{2}u - 1)^2}. \tag{8}$$

With the restriction $\hbar\omega \ll kT$, Eq. (4) transforms into

$$u = 2\bar{\varepsilon}^2 = 2\left(\frac{\hbar}{2I\omega} + \frac{kT}{I\omega^2}\right), \tag{9}$$

*) The principal polarization — as obtained when the depolarization resulting from the torsional vibrations is taken into account — is termed pseudoprincipal polarization.

and, ultimately, Eq. (8) assumes the form

$$\frac{1}{P'_p}-\frac{1}{3}=\left(\frac{1}{P_p}-\frac{1}{3}\right)\cdot\frac{1}{(B\cdot T-C)^2} \tag{10}$$

with

$$B=\frac{3k}{I\omega^2}, \qquad C=1-\frac{3\hbar}{2I\omega}. \tag{11}$$

By Eq. (10), which yields the dependence of the pseudoprincipal polarization on temperature, it will be seen that the value of the pseudoprincipal polarization is the greater the lower that of the principal polarization and the higher that of the quantity B (i. e., the lesser $I\omega^2$). Thus, the effect under consideration should attain its greatest value in case of benzene (if the virtual electronic oscillator is assumed to be plane and symmetrical here, whence $P_p=\frac{1}{7}$).

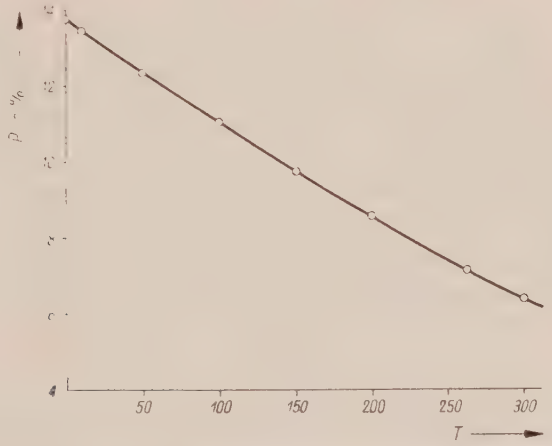

Fig. 1. Degree of pseudoprincipal polarization of the fluorescence of benzene in glycerol *versus* the temperature T
o — points computed with Eqs. (10) and (11)

Fig. 1 shows a graph of the degree of pseudoprincipal polarization *versus* the temperature, as given by Eq. (10), for the fluorescence of a solution of benzene in glycerol. The value of the moment of inertia $I=1.55\times10^{-38}$ g/cm² has been taken from [7], and that of the frequency of the torsional vibrations $\omega=5\times10^{12}s^{-1}$ computed from (10) and (11) assuming $P_p=\frac{1}{7}$ and $P'_p=\frac{1}{11\frac{1}{4}}$ (at $T=263°$K) as resulting from measurements by Feofilov [8] of the pseudoprincipal polarization for benzene in glycerol solution. The frequency of the torsional vibrations as computed from (3) and (4) is $4.8\times10^{12}s^{-1}$, whilst that resulting from (5) (at the same values of I, P_p and P'_p) is $5.4\times10^{12}s^{-1}$. The range of applicability of formula (10) is restricted, due to the condition $u/2=\overline{\varepsilon^2}\ll1$, since $\overline{\sin^2\theta}$ has been replaced by $\overline{\theta^2}=u/2$ (viz. [3]); thus, Eq. (9) should yield a good approximation at low temperatures. On the other hand,

the approximation (9) was obtained on the assumption $\hbar\omega \ll kT$, thus applying to high temperatures. However, if Eq. (9) is used for low temperatures, no considerable error arises, since then the fraction $\frac{1}{e^{\hbar\omega/kT}-1}$ assumes a small value. It should be stressed that Eqs. (2), (3) and (4), and, *a fortiori*, Eqs. (10) and (5) do not hold for large values of the depolarization, since it is only for very small amplitudes that the torsional vibrations about the three principle axes of inertia may be considered to be mutually independent. The assumption of a constant frequency ω of the torsional vibrations is also a simplification, as ω varies with the temperature *).

With the restriction $u \ll \frac{2}{3}$, Eq. (8) yields

$$\frac{1}{P'_p} \cong \frac{1}{3} + \left(\frac{1}{P_p} - \frac{1}{3}\right) \cdot (1+3u), \tag{12}$$

where

$$P'_p = \frac{P_p}{1+(3-P_p)u} \cong P_p[1-(3-P_p)u] \quad \text{for} \quad u \ll \frac{1}{3-P_p}. \tag{13}$$

The substitution of Eq. (9) in (13) yields Eq. (5), as obtained previously by A. Jabłoński [6].

It is noteworthy that Eqs. (8), (10), and especially (12) derived in the present paper are of a shape resembling that of Perrin's formula accounting for rotational depolarization [9]. On introducing the emission anisotropy r **) according to A. Jabłoński [10] in place of the degree of polarization P, Eq. (10) reduces to

$$\sqrt{r'_p} = \sqrt{r_p}(C - B \cdot T), \tag{14}$$

where r'_p and r_p denote the emission anisotropy corresponding to the pseudoprincipal and principal polarizations, respectively.

The author wishes to express his sincere thanks to Professor A. Jabłoński for his valuable advice and remarks, and is indebted to Professor W. Mościcki, Director of the Departement of Physics, Technical University, Gdańsk, where the present paper was prepared, for his unremitting interest.

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*) The wave numbers of the torsional vibrations, as obtained by A. Fruhling [7], who investigated the external Raman effect in pure benzene crystals, vary from $\tilde{\nu} = 35$, 63 and 105 cm^{-1} at $T = 273°$K to $\tilde{\nu} = 53$, 74 and 117 cm^{-1} at $T = 188°$K.

**) For polarized exciting light

$$r = \frac{I_{||} - I_{\perp}}{I_{||} + 2I_{\perp}} = \frac{2P}{3-P},$$

where $I_{||}$ and $I_{\perp}$ denote the component of the emitted intensity parallel and perpendicular to the electric vector of the exciting light, respectively.

REFERENCES

[1] M. Makowiecka, Acta Phys. Pol., **2** (1934), 361.
[2] F. Perrin, C. R. Acad. Sci., Paris, **180** (1925), 581, ibid., **182** (1926a), 219. J. Phys. Radium **7** (1926b), 390.
[3] A. Jabłoński, Acta Phys. Pol. **10** (1950), 33, ibid., **10** (1950), 193.
[4] — Z. Phys. **96** (1935), 236.
[5] — Acta Phys. Pol. **4** (1935), 371, 389, ibid., **5** (1936), 271.
[6] — Nuovo Cimento **2** (1955), 995.
[7] A. Fruhling, Ann. de Phys. **6** (1951), 401.
[8] P. P. Feofilov, C. R. Acad. Sci. URSS **57** (1947), 343.
[9] F. Perrin, Ann. de Phys. **12** (1929), 169.
[10] A. Jabłoński, Acta Phys. Pol. **16** (1957), 471.

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EXPERIMENTAL PHYSICS

Note on the Theory of the Self-depolarization of Photoluminescence of Solutions

by

C. BOJARSKI

Presented by A. JABŁOŃSKI on March 12, 1958

The effect of depolarization of the fluorescence of solutions accompanying a rise of concentration c of the solute was discovered simultaneously by a number of investigators in 1924 [1]. Feofilov and Sveshnikov [2] investigated i. a. the dependence of the depolarization of fluorescence on concentration for a number of compounds, and gave the following experimental relation:

$$\frac{1}{p} = \frac{1}{p_0} + Ac\tau\,, \tag{1}$$

where p denotes the degree of polarization at a given concentration c of the dye, p_0 — the degree of polarization as the concentration tends towards zero, τ — the lifetime of the excited state, and A — a constant. The theory of concentrational depolarization was first given by Vavilov [3] and Förster [4]. Both theories yield the linear dependence (1) at low concentrations. Subsequently, Weber [5] obtained an expression accounting for the depolarization of the fluorescence of solutions and showing the reciprocal of the degree of polarization to be a linear function of the concentration and thus to have the form of the empirical Eq. (1).

Recently a general theory of the self-depolarization of the photoluminescence of solutions was given by Jabłoński [6], [10]. It can be shown that the general expression derived therein, which describes the dependence of the degree of polarization on the concentration of the luminescent molecules, may be presented in a simpler form, similar to that of Eq. (1), valid, however, for both low and high concentrations.

Jabłoński's theory of self-depolarization of luminescence, which accounts for the fluctuations of the concentration of the luminescent molecules, introduces the simplified model of a luminescent centre as consisting of one primarily excited luminescent molecule surrounded by what

is termed the active sphere, which may contain non-excited molecules of the same kind.

With regard to the number of luminescent molecules present within an active sphere, the latter are divided into groups as follows: a centre is said to belong to the k-th group if it contains one excited and $k-1$ non-excited molecules. The excitation energy may be transferred from one luminescent molecule to another only if both are contained within one and the same centre; the migration of the excitation energy may take place several times between such molecules. The probability of the energy transfer is assumed proprotional to the time and equal for all non-excited molecules belonging to a given centre. Each group of centres is now split into two sub-groups. Sub-group A contains the centres wherein the excitation energy is present in the molecule primarily excited by absorption of polarized light; sub-group B contains the centres wherein the excitation energy is attached to one of the molecules not originally excited. Generally, the light emitted by centres of sub-group A is polarized, whereas that emitted by centres of sub-group B is totally depolarized. The latter fact is explained by the isotropic orientational distribution of the molecules excited in the course of the migration of the excitation energy. Basing on this model of a luminescent centre, Jabłoński, in his general theory of self-depolarization of the photoluminescence of solutions, obtained the following expression for the dependence of the degree of polarization on the concentration of the luminescent molecules, if self-quenching is neglected:

$$P = 3P_0 \frac{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \cdot \frac{2}{k+1}}{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \cdot \left[(3-P_0) + \frac{2P_0}{k+1}\right]}. \tag{2}$$

Here, P_0 denotes the fundamental degree of polarization, and

$$\nu = v \cdot n, \tag{3}$$

where v is the volume of the active sphere, and n — the number of luminescent molecules per cm³.

In order to obtain Eq. (2) in a more simple form, the following notation is introduced:

$$\varphi(\nu) = \sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \cdot \frac{2}{k+1} \quad \text{and} \quad \psi(\nu) = \sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!}; \tag{4}$$

thus,

$$P = 3P_0 \cdot \frac{\varphi(\nu)}{(3-P_0) \cdot \psi(\nu) + P_0 \cdot \varphi(\nu)} \tag{5}$$

or

$$\frac{1}{P}-\frac{1}{3}=\left(\frac{1}{P_0}-\frac{1}{3}\right)\cdot f(\nu)\,, \tag{6}$$

where

$$f(\nu)=\frac{\psi(\nu)}{\varphi(\nu)}\,. \tag{7}$$

The functions $\psi(\nu)$ and $\varphi(\nu)$ are obtained in finite form by performing the summations in (4), which yield

$$\psi(\nu)=e^{\nu} \quad \text{and} \quad \varphi(\nu)=2\left[\left(\frac{1}{\nu}-\frac{1}{\nu^2}\right)e^{\nu}+\frac{1}{\nu^2}\right]. \tag{8}$$

Substituting (8) into (7) yields

$$f(\nu)=\frac{\nu^2}{2[\nu-1+e^{-\nu}]}\,. \tag{9}$$

Thus, Eq. (6) is now obtained as follows:

$$\frac{1}{P}-\frac{1}{3}=\left(\frac{1}{P_0}-\frac{1}{3}\right)\frac{\nu^2}{2[\nu-1+e^{-\nu}]}\,; \tag{10}$$

a form similar to that of Perrin's rotational depolarization formula [7]. The shape of the dependence of the degree of polarization P on ν, and thus on concentration c, is determined by function $f(\nu)$. It should be noted that, if $\nu\to 0$, $f(\nu)\to 1$, and thus $P\to P_0$, whereas if $\nu\to\infty$, $f(\nu)\to\infty$ and $P\to 0$. Expanding $f(\nu)$ in a MacLaurin series and neglecting higher order terms yields

$$f(\nu)\cong 1+\tfrac{1}{3}\nu \tag{11}$$

which reduces (10) to

$$\frac{1}{P}-\frac{1}{3}\cong\left(\frac{1}{P_0}-\frac{1}{3}\right)\left(1+\frac{1}{3}\nu\right), \tag{12}$$

a form identical with that given by Jabłoński, who derived it directly from Eq. (2) neglecting all terms but the first two. The linear function $1+\frac{1}{3}\nu$ is found to yield an excellent approximation of $f(\nu)$, as defined by (9), for values of $\nu<0.3$ (see Fig. 2). The maximum difference in the $f(\nu)$ values as calculated between Eqs. (9) and (11) does not exceed 0.004, and the difference in the degree of polarization P, as calculated from Eqs. (10) and (12), is less than 0.1%, thus remaining within the limit of experimental error. If the function

$$f^0(\nu)=f(\nu)-1 \tag{13}$$

is substituted for $f(\nu)$, Eq. (6) takes the more convenient form

$$\frac{1}{P}=\frac{1}{P_0}+\left(\frac{1}{P_0}-\frac{1}{3}\right)\cdot f^0(\nu)\,. \tag{14}$$

It will be seen from (14) that the shape of the dependence of $1/P$ on v is the same for various dyes exhibiting concentrational depolarization; any possible differences consist in a translation and rotation of the $f^0(v)$ curve, augmenting towards lower values of P_0. Graphs of the function $f^0(v)$ are given in Figs. 1 and 2. It will be seen from Fig. 1 that

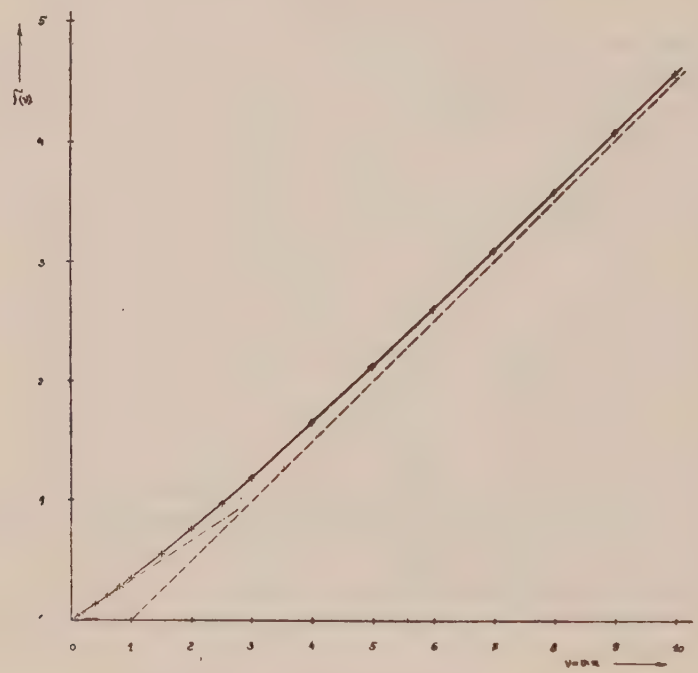

Fig. 1. Graph of $f^0(v)$ *versus* $v = v.n.$
+ — calculated from (13) and (9)
—· ·—· - $\frac{1}{3}v$
— — — — asymptote $\frac{1}{2}v - \frac{1}{2}$

for values of $v < 0.3$, the line $\frac{1}{3}v$ is a good approximation of the function $f^0(v)$; for very high values of v, the latter is approximated by the asymptote $\frac{1}{2}v - \frac{1}{2}$. For values of v the curvature of the function $f^0(v)$ is insignificant within the interval $\langle 0.3; 3\rangle$, after which it is once more almost linear. This makes it possible to co-ordinate the linear function $G(v)$ given by

$$G(v) = 0.48 \cdot v - 0.27 \tag{15}$$

to the function $f^0(v)$ within the interval $3 < v < 10$. The error inherent in the value ΔP of the degree of polarization originating in the approximation used does not exceed 0.25%; the difference $f^0(v) - G(v)$ does not exceed 0.025.

In order to compare the graphs calculated from the theory with experimental data it is necessary to evaluate the volume of the active

sphere v, which appears in (10) through the relation $\nu = vn$. The volume of the active sphere v is determined from experimental data as follows: let n_k denote the number of molecules per cm³, P_k — the corresponding degree of polarization experimentally obtained, and P_0 — the fundamental degree of polarization; then, with Eq. (6), the value of $f(\nu_k)$ is computed. Once the value of $f^0(\nu_k)$ is known, that of ν_k may be taken from graphs (1) and (2) or, if $\nu_k > 3$, from the relation

$$f^0(\nu_k) \cong G(\nu_k) = 0.48 \cdot \nu_k - 0.27.$$

With ν_k and n_k, the values of v_k corresponding to the various concentrations are obtained from $v_k = \frac{\nu_k}{n_k}$; finally, the mean value v is computed.

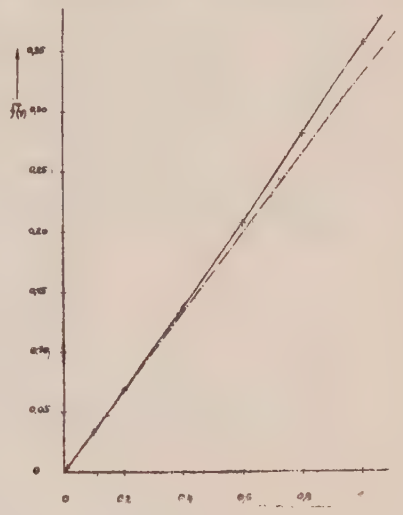

Fig. 2. Graph of $f^0(\nu)$ *versus* $\nu = v.n.$ for $\nu < 1$.
+ — calculated from (13) and (9)
.—.—. $\frac{1}{3}\nu$

Eq. (14) may be compared with the experimental results obtained by Feofilov and Sveshnikov when investigating concentrational depolarization of fluorescein and rhodamin B in anhydrous glycerine. In Figs. 3 and 4 the continuous curve represents the theoretical relation as evaluated

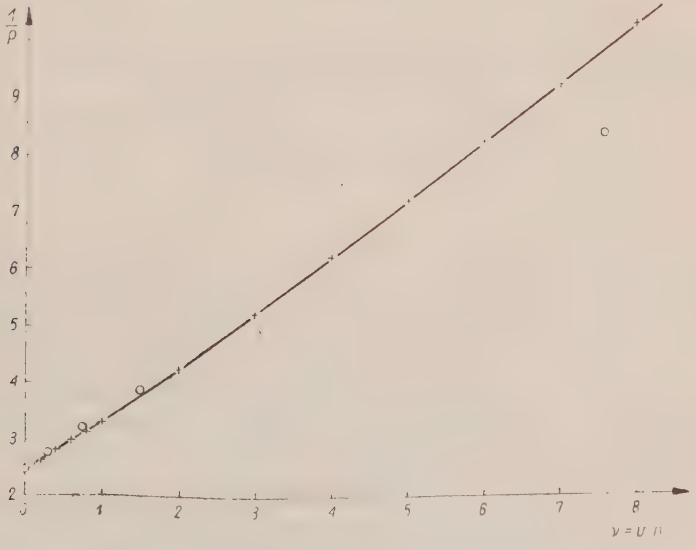

Fig. 3. Reciprocal of the degree of polarization of the fluorescence of fluorescein in glycerol *versus* $\nu = v.n.$
o — experimental results of Feofilov and Sveshnikov
+ — values calculated from (14)

from (14), whereas the experimental points are represented by small circles. For entering the experimental points in the diagram, use has been made of the tables in Vavilov's paper [3], which contain the numerical results of the measurements by Feofilov and Sveshnikov. In order to determine the values of v corresponding to the various values of n, the latter had to be multiplied by the volume of the active sphere v which had been determined as described above. The mean values of the

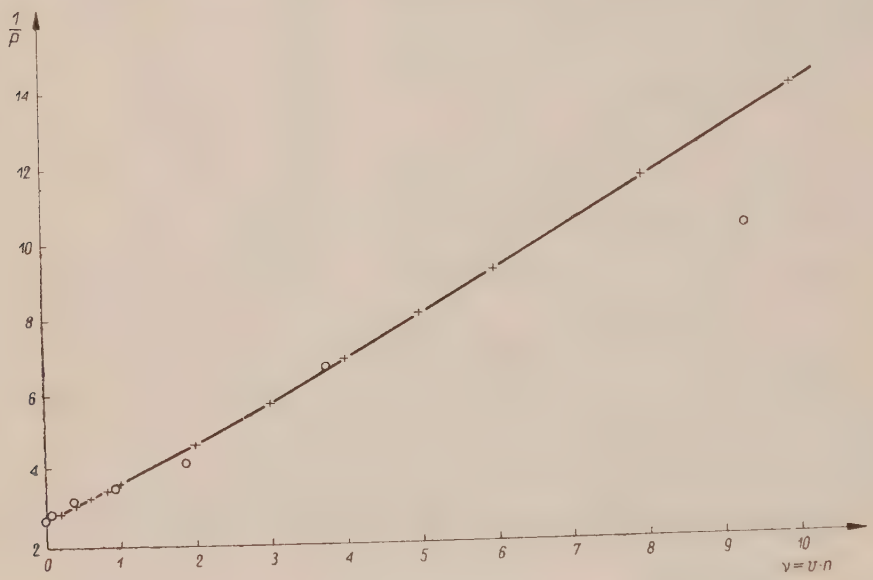

Fig. 4. Reciprocal of the degree of polarization of the fluorescence of rhodamine B in glycerol *versus* $v = v.n$.

○ — experimental results of Feofilov and Sveshnikov

\+ — values calculated from Eq. (14)

radii of the active spheres are 76.8 Å and 89.5 Å for fluorescein and rhodamin B, respectively. It is noteworthy that the radii of the active spheres of both dyes are practically independent of the concentration within the range of values investigated, as already stated [8]. It will be seen from Figs. 3 and 4 that the experimental results are in good agreement with the theory for not too high concentrations of solution (see also [9]). The last experimental point on the graphs 3 and 4 deviates somewhat from the calculated curve. This insignificant deviation is due probably to the effect of the self-quenching, which was neglected by derivation of Eqs. (2) and (10). Thus, Jabłoński's theory yields a linear dependence of the type of the empirical Eq. (1), with this difference, however, that it predicts a lesser steepness for low concentrations and a greater steepness for high concentrations.

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REFERENCES

[1] E. Gaviola and P. Pringsheim, Zt. f. Phys. **24** (1924), 24; F. Weigert and G. Käppler, ibid., **25** (1924), 99; V. L. Levshin, ibid., **26** (1924), 274.

[2] P. P. Feofilov and B. Sveshnikov, J. exp. theor. Phys. **10** (1940), 1372.

[3] S. I. Vavilov, J. Phys. USSR **7** (1943), 141.

[4] Th. Förster, Ann. Phys. **2** (1948), 55.

[5] G. Weber, Trans. Faraday Soc. **50** (1954), 552.

[6] A. Jabłoński, Acta Phys. Pol. **14** (1955), 295; ibid. **13** (1954), 175.

[7] F. Perrin, Ann. de Phys. **12** (1929), 169.

[8] M. Frąckowiak, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 809.

[9] A. Kawski, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. **6** (1958), 533.

[10] A. Jabłoński, Acta Phys. Polon. (1958) (in press).

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 11

В работе рассматривается уравнение вида

$$(*) \qquad \frac{dx}{dt} = A(t)x + f(tx),$$

где $A(t)$—замкнутый линейный оператор определенный на линейном подмножестве пространства Банаха E. Значения оператора $A(t)$ принадлежат к E. При помощи метода дифференциальных неравенств доказывается ряд теорем об оценках норм решений через максимальные интегралы обыкновенных уравнений. Существенным образом используется эпидермическая теорема Важевского. Доказывается теорема об единственности, соответствующая теореме Камке. Рассматриваются обобщенные решения, являющиеся решениями интегральных уравнений. Для обобщенных решений приводятся оценки, полученные методом интегральных неравенств. Примером применений теорем об оценках являются нелокальные обобщения теорем о существовании, доказанных в локальном случае Красносельским, Крейном и Соболевским.

В конце работы приводятся достаточные условия существования производной решения относительно параметра.

Пусть $L_{ik} \geqslant 0$.

Доказывается, что нижеприведенные условия

(a_λ) $L_{ii} < \lambda$, $i = 1, \ldots, n$, и $|\lambda\delta_{r_i r_k} - L_{r_i r_k}|_1^p > 0$ для всех систем индексов $1 \leqslant r_1 < r_2 < \ldots < r_p \leqslant (n+p)/2$, где $n \geqslant p \equiv n \pmod 2$,

(b_λ) все главные миноры матрицы $\|\lambda\delta_{ik} - L_{ik}\|_1^n$ — положительные;

(*c*) для произвольного преобразования $y_i = f_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, n$ картезианского произведения E метрических совершенных пространств $E_1, E_2, \dots, E_n$ в E такого, что

$$d_i(y_i', y_i'') \leqslant \sum_{k=1}^{n} L_{ik} d_k(x_k', x_k''), \qquad i = 1, 2, \dots, n$$

для любых пар точек пространства E, где d_i обозначает расстояние в E_1, существует инвариантная точка,

связаны следующими реляциями: $(a_\lambda) \to (b_\lambda)$, $(c) \to (b_1)$, $(b_1) \to (c)$, причем в этом последнем случае существует лишь одна инвариантная точка, которая может быть получена методом последовательных приближений, исходя из любой точки пространства E.

Т. ЛЕЖАНСКИЙ, ФОРМУЛА НА СПЕКТРАЛЬНОЕ РАЗЛОЖЕНИЕ . . . . . стр. 689—690

Через H обозначим пространство Гильберта; через A — симметрическую, определенную на H операцию, которой спектр содержится в конечном промежутке $\Delta = (a, b)$; через E_λ — спектральное семейство операции A.

В работе дается эффективная формула для абстрактной функции $E_\lambda(z^0)$, где z^0 — произвольный, определенный элемент пространства H. Пусть $\mathcal{H}$ будет гильбертово пространство определенных на Δ функций $x(\lambda)$ со значениями в H. Полагаются: 1°. Система псевдонорм в $\mathcal{H}$, 2°. $\mathfrak{X}$ — распространение пространства $\mathcal{H}$ при системе псевдонорм, 3°. $\mathcal{A}$ — симметрическая и ограниченная операция в $\mathcal{H}$, $x^0(\lambda)$ — некоторый элемент из $\mathcal{H}$. Тогда

$$E_\lambda(z^0) = \int_0^\infty \int_a^\lambda e^{-s\mathcal{A}}(x^0(\mu))\, d\mu\, ds\,.$$

Бесконечный интеграл понимается в смысле сходимости в пространстве $\mathfrak{X}$.

ЯН МИКУСИНСКИЙ, О ФУНКЦИИ ЯВЛЯЮЩЕЙСЯ РЕЗУЛЬТАТОМ ПРЕОБРАЗОВАНИЯ ЛАПЛАСА ФУНКЦИИ e^{-s^α} . . . . . стр. 691—693

Функция

$$F_\alpha(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-tz - z^\alpha} dz \qquad (0 < \alpha < 1,\ 0 < t < \infty)$$

может быть представлена в виде интеграла

$$F_\alpha(t) = \frac{1}{\pi} \frac{\alpha}{1-\alpha} \frac{1}{t} \int_0^\pi u e^{-u} d\varphi\,,$$

где

$$u = t^{-\frac{\alpha}{1-\alpha}} \left(\frac{\sin \alpha\varphi}{\sin \varphi}\right)^{\frac{\alpha}{1-\alpha}} \frac{\sin(1-\alpha)\varphi}{\sin \varphi}\,.$$

Полученная автором новая формула полезна тем, что под знаком интеграла находится действительная и положительная функция. Отсюда можно вывести асимптотические формулы

$$F_\alpha(t) \sim K t^{-\frac{2-\alpha}{2-2\alpha}} \exp\left(-A t^{-\frac{\alpha}{1-\alpha}}\right) \quad \text{для} \quad t \to 0$$

и

$$F_\alpha(t) \sim M t^{-1-\alpha} \quad \text{для} \quad t \to \infty$$

с постоянными

$$A = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}, \quad K = \frac{1}{\sqrt{2\pi(1-\alpha)}}\,\alpha^{\frac{1}{2-2\alpha}}, \quad M = \frac{\sin \alpha\pi}{\pi}\,\Gamma(1+\alpha).$$

Пусть m — Банахово пространство всех ограниченных действительных последовательностей $x = (a_n)$ с нормой $\|x\| = \sup\limits_n |a_n|$. M — Банахово пространство всех классов эквивалентности действительных измеримых по Лебегу функций $x(0)$, определенных на отрезке $\langle 0,1\rangle$ с нормой $\|x\| = \sup\limits_{0\leqslant t\leqslant 1} \operatorname{ess} |x(t)|$.

В настоящей заметке доказано, что эти пространства изоморфны.

Методом последовательных приближений приводится решение сингулярного интегрального уравнения

$$u(t) = \lambda \int\limits_L \frac{K[t, \tau, u(\tau)]}{\tau - t}\, dt,$$

где L — множество p незамкнутых и непересекающихся дуг на комплексной плоскости, а интеграл понимается как главное значение (в смысле Коши).

Предполагается, что

1) функция $K(t, \tau, \xi)$, определена для $t, \tau \in L$, $\xi \in X$ (где X — произвольное пространство Банаха) удовлетворяет неравенствам:

$$\|K(t, \tau, \xi)\|_X \leqslant k' \|\xi\|^r + k''$$

$$\|K(t, \tau, \xi) - K(t_1, \tau_1, \xi_1)\|_X \leqslant k\left[|t-t_1|^\nu + |\tau-\tau_1|^\mu + \|\xi-\xi_1\|_X\right]$$

$$\left(K, K', K'' > 0; \quad 0 < \mu < \nu \leqslant 1; \quad 0 \leqslant r \leqslant \frac{\beta}{\alpha} \leqslant 1\right),$$

2) функция $K(t, \tau, \xi)$ принадлежит к классу $\mathcal{H}_{\mu,\nu}$.

Целью настоящей работы была проверка зависимости основной поляризации от температуры.

Констатировано, что степень основной поляризации антрацена в жестком растворе (полиметакрилат метила) для температуры около —150°C равняется 27,8%, тогда как для комнатной температуры 26,6%. Полученные данные экспериментально подтверждают теоретические предположения А. Яблонского, касающиеся влияния торсионных колебаний на основную поляризацию.

На основании результатов теории А. Яблонского, касающейся влияния торсионных колебаний люминесцирующих молекул на основную поляризацию фотолюминесценции растворов, приводится выражение на зависимость основной поляризации от температуры, а также проводится дискуссия относительно полученной зависимости.

Доказано, что полученное А. Яблонским общее выражение, описывающее зависимость поляризации фотолюминесценции от концентрации флуоресцирующего вещества, для случая, в котором самотушением можно пренебречь, может быт представлено в простом виде.

Проведена дискуссия относительно полученной зависимости, а также констатировано, что экспериментальные результаты Феофилова и Свешникова можно вполне хорошо описать, применяя теорию А. Яблонского.

TABLE DES MATIÈRES

Mathématique

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Cena zł 20.—